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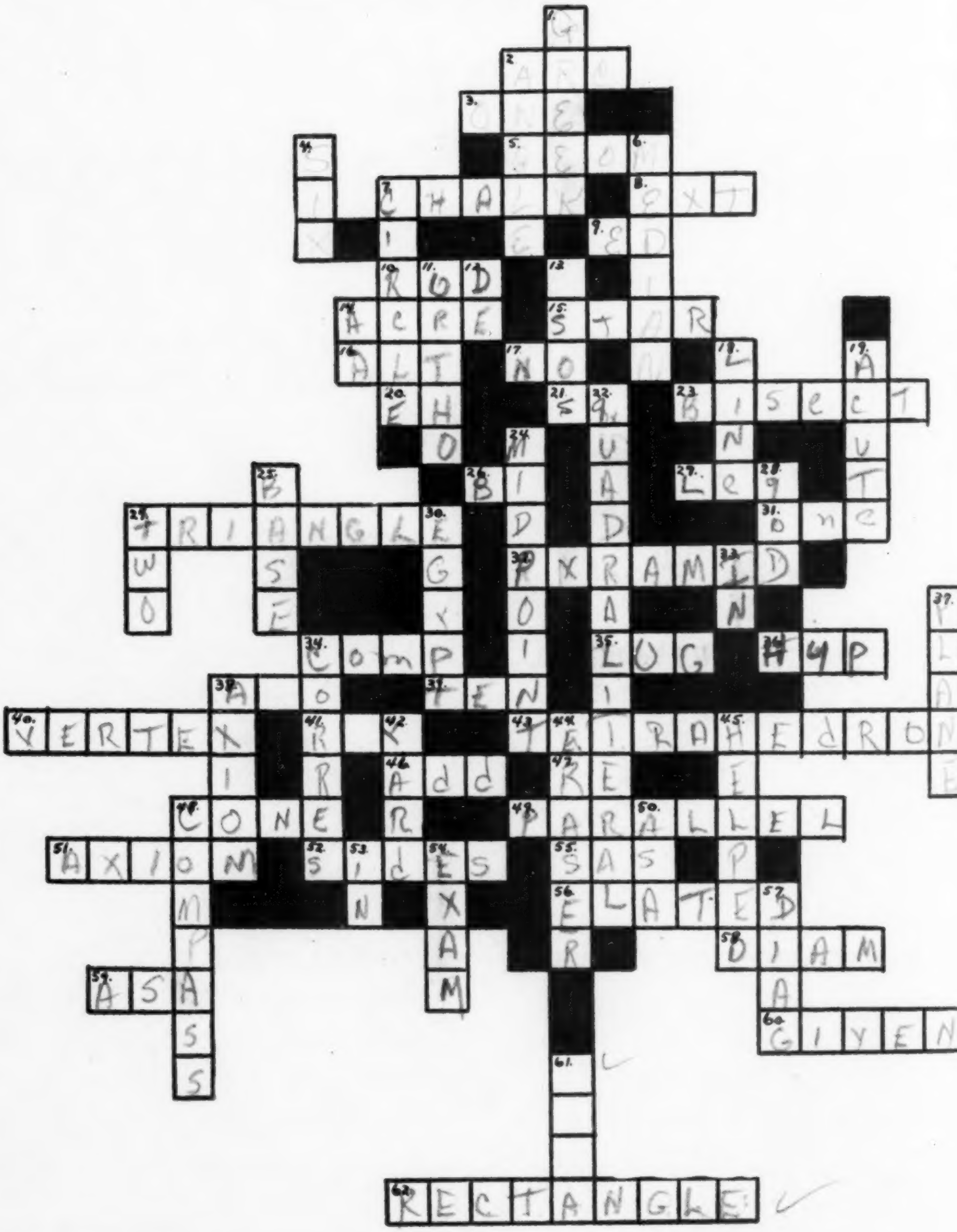

* See page 372 for key.

HORIZONTAL

2. A part of the circumference of a circle.
3. Number of lines which can be drawn through a point parallel to a given line.
5. Branch of mathematics (abbreviation)
7. Material often used in drawing mathematical figures in a schoolroom.
8. Name of angle formed between one side of a triangle and the extension of another side. (abbreviation)
9. First and last letters of the name of the famous Greek mathematician who compiled the first geometry textbook.
10. Unit of measurement.
14. A measure of area with which farmers are especially familiar.
15. A geometric figure which is frequently displayed at Christmas time.
16. The perpendicular distance from any vertex of a triangle to the opposite side. (abbreviation)
17. Abbreviation of the word "number."
20. First and fourth letters of one of the digits.
21. An equilateral quadrilateral. (abbreviation)
23. To divide into two equal parts.
26. A prefix meaning "two."
27. The side opposite an acute angle in a right triangle.
29. A three-sided polygon.
31. A small number.
32. A geometrical figure in which tombs of Egyptian kings have been found.
34. Two angles which form a right angle are are said to be __________. (abbreviation)
35. An abbreviation of a term often used in an advanced study of exponents.
36. The longest side of a right triangle. (abbreviation)
38. The first three letters of the name of a perpendicular from the center of a regular polygon to one of its sides.
39. A two-digit number.
40. The point at which two rays meet to form an angle.
41. A line extending in only one direction from a fixed point.
43. One of the regular polyhedrons.
46. To find a sum.
47. First two letters of a special quadrilateral.
48. A mathematical figure often served in edible form.
49. Two lines in the same plane, which do not meet no matter how far they are extended, are said to be __________.
51. A general statement assumed to be true.
52. A rhombus has four equal ones.
55. A method of proving triangles congruent. (abbreviation)
56. The way a geometry student feels when he has completed a difficult problem.
58. The longest chord that can be drawn in a circle. (abbreviation)
59. An abbreviation of the following theorem: "Two triangles are congruent if one side, the opposite angle, and another angle of one triangle are equal respectively to one side, the opposite angle, and another angle of the second triangle."
60. The hypothesis contains this information.
62. A special quadrilateral.

VERTICAL

1. People of this nationality contributed a great deal to our knowledge of geometry.
2. The figure formed when two rays meet.
4. Number of sides in a hexagon.
6. A line drawn from any vertex of a triangle to the midpoint of the opposite side.
7. The geometric figure which a sunflower suggests.
11. The "center" at which the altitudes of a triangle are concurrent.
12. First two letters of a term used as a measure of angles.
13. If a triangle has at least two sides equal, it is said to be __________. (abbreviation)
14. First and last letters of a branch of mathematics.
18. The shortest distance between two points is a straight __________.
19. An angle containing less than ninety degrees is said to be __________.
22. A closed four-sided figure.
24. The point located when a line is bisected.
25. The __________ angles of an isosceles triangle are equal.
28. "The laws of nature are but the mathematical thoughts of __________."
29. The number of equal parts into which an angle is divided when it is bisected.
30. A country in which geometry was studied hundreds of years ago.
33. An abbreviation for a unit of measurement.
34. __________ sides and angles of congruent triangles are equal. (abbreviation)
37. A surface such that a straight line connecting any two points in it will lie completely in that surface.
38. The statement "The whole equals the sum of its parts" is known as an __________.
42. A unit of measurement.
44. A school supply often used by students of mathematics.
45. Sometimes a mathematics students needs to be __________ by the teacher.
48. An instrument used by geometry students.
50. A method of proving triangles congruent. (abbreviation)
53. The hypothesis of a theorem often begins with this word.
54. A device used by teachers to test students' knowledge.
57. A line connecting opposite corners of a parallelogram, for example, is called a __________. (abbreviation)
61. A regular hexahedron.

THE MATHEMATICS TEACHER

Volume XXXVII Number 8

Edited by William David Reeve

Psychological Points of View on the Teaching of Arithmetic

By Dr. H. V. Baravalle
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The psychological point of view focuses primarily on the effect which the learning of a subject has on the mind itself. It seeks to find out the creative impulses which can be conveyed, for instance, through the lessons of arithmetic during the various grades.

Already with children at an early age, even before they enter school, phenomena due to creative contact with numbers can be observed. When the child begins to count he becomes aware of the fact that his use of the words one, two, three, four etc. is different from his usual speech. They have their regular sequence. The word "one" always comes first in counting. The word "two" has to wait until one has passed and "three" must wait even longer, etc. Each one of these little words is preceded by another one and in its turn it is the key word for the following one. If a child is given the opportunity to count out loud repeatedly, the impression of this sequence of words is strengthened and a joyful response is awakened. The fact that the child hears himself speak connectedly in counting calls forth the satisfaction of moving within an ordered sequence. This satisfaction does not derive from indulging in subjective arbitrariness, but from partaking in lawfulness.

From simple counting arithmetic lessons can proceed in the first grade to rhythmic counting, accentuating some numbers in contrast to others. This can be done by speaking them louder or by pronouncing them more distinctly and slowly in comparison to others passed over more quickly. As an example accent every third number when counting and the sequence obtained is:

$$1\ 2\ \underline{3}\quad 4\ 5\ \underline{6}\quad 7\ 8\ \underline{9}\quad 10\ 11\ \underline{12}\ \cdots$$

By accentuating every fourth number the following sequence results:

$$1\ 2\ 3\ \underline{4}\quad 5\ 6\ 7\ \underline{8}\quad 9\ 10\ 11\ \underline{12}\quad 13\ 14\ 15\ \underline{16}\ \cdots$$

Multiplication tables are obtained by making a choice among the complete sequence of numbers, that is dropping some and retaining others in rhythmic order. To the child's first impression of the set order in numbers there now comes an additional one, that of making regulated choices from their original sequence. In connection with these exercises the pupils can walk or clap their hands in rhythm with the counting, or perform any similar motion. Introducing the element of rhythm into arithmetic brings it close to experiences the child has had in music.

In the historical development of mathe-

matics during the Greek period mathematical procedures of a rhythmic nature were already applied. Eratosthenes, for instance, when determining the prime numbers went repeatedly over the regular sequence of numbers and eliminated first every second number that followed 2, thus:

1 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9 ~~10~~ 11 ~~12~~ 13 ~~14~~ 15 · · ·

Then he eliminated also every third number that followed 3, thus:

1 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~ 13 ~~14~~ ~~15~~ · · ·

When taking up four its whole row is found already eliminated. By crossing out every fifth number that follows 5 some numbers like 10, 15 and 20 are found already eliminated. The first number remaining to be crossed out is 25. The numbers that remain after the whole procedure has been completed are the prime numbers.

Another view on arithmetic from the psychological angle is derived from the very nature of numbers. Numbers lead the activity of the mind beyond immediate perception. We do not perceive numbers as sense phenomena, as we do colors, sounds, etc.; numbers arise in our mind in addition to sense observation. This fact becomes apparent when we recognize the motive behind an impetus to count. The leaves on a branch of a tree, for instance, would not excite in us a desire to count them—the result obtained would not be worth the effort. The case is different with the petals of a flower, for their number is often characteristic of the blossom and determines its shape. What makes it reasonable to count in one case and not in the other is the existence or lack of a relationship between the elements that we count. This is recognized in the fact that apples and pears can be added only after a relationship between apples and pears has been established under the common concept "fruit."

The process of counting can be studied by emptying a basket of apples on a table. Involuntarily with the hand, or only in our thoughts, we arrange them in groups and count them as 3+4+2 or 3+3+3. Nine objects, lying about on the table, can then be counted at the first glance. In the following the 9 points are arranged first without order, then as 3×3 in a square:

9 points in broken order

9 points arranged in a square

9 points arranged in a square standing on end

The successive horizontal lines contain in the square standing on end 1+2+3+2+1 points. Similarly four times four points lead to the following result:

16 points arranged in a square

16 points arranged in a square standing on end

$$16=1+2+3+4+3+2+1$$

and 5 times 5 points:

$$25=1+2+3+4+5+4+3+2+1$$

Summing up the successive square numbers one obtains:

1×1=	1
2×2=	1+2+1
3×3=	1+2+3+2+1
4×4=	1+2+3+4+3+2+1
5×5=	1+2+3+4+5+4+3+2+1
. . . . .	. . . .

In each row the numbers increase up to the middle and then decrease symmetri-

cally. The square of every number represents the sum of the arithmetical progression of numbers going from one up to the original number and back again to one. Every subsequent line has 2 numbers more than the preceding one; from line to line a higher middle number is added and the number preceding it is repeated. The new numbers thus added are:

$$\begin{array}{l}
1+0=1\\
2+1=3\\
3+2=5\\
4+3=7\\
5+4=9\\
6+5=11\\
7+6=13\\
8+7=15\\
\cdots
\end{array}$$

The odd numbers manifest themselves as the differences between the square numbers:

$$\begin{array}{r}
1-0=1\\
4-1=3\\
9-4=5\\
16-9=7\\
25-16=9\\
36-25=11\\
49-36=13\\
64-49=15
\end{array}$$

Many number relationships can be demonstrated to arouse the pupil's interest in mathematical facts. They appeal no less to the mind than experiences of color and form and hold the additional stimulus that they can be revealed without any other tool than one's own mental activity.

In working with a class special occasions may be seized to display numerical facts. On a certain day one of the children may have his eighth birthday. His little brother at home may then be 2 years old. The child therefore lived four times as long as the little brother. One may further call attention to the fact that in the next year the child will have his ninth birthday and the little brother his third. The age ratio between them has changed; the child has now three times the little brother's age. A year later the child will be ten and the little brother four, then eleven and the little brother five, the next year twelve and the brother six. At this point the child's life will be twice that of the brother. This diminished ratio will make the child think and it can be pointed out to him that this ratio also becomes manifest in life. At the time when the child is eight and has four times the age of the little brother, the age difference between the two children is apparent to every one on sight. But once they will have reached 42 and 48 it might possibly be doubtful to recognize which was the older of the two.

An important factor from a psychological point of view is the continuity in the thoughts pursued during the lesson. If the problems introduced in a lesson deal, for instance, first with stretches traversed by trains, then with pipe lines that fill a container and afterwards with sales prices of fruit and vegetables, etc., the continuity of thought is disrupted each time. By continuing a line of thought greater interest can be aroused both in verbal and numerical problems. For instance, after taking up $12=5+7$, one can proceed to elaborate further on the problem by giving other compositions of this number, thus:

$$\begin{array}{ll}
12=1+11 & 12=7+5\\
12=2+10 & 12=8+4\\
12=3+9 & 12=9+3\\
12=4+8 & 12=10+2\\
12=5+7 & 12=11+1\\
12=6+6 &
\end{array}$$

The first numbers to the right of the equal sign increase step by step from one to eleven, while those in the next column diminish from eleven to one. The amount of possible compositions is eleven, one less than the original number 12. Continuing this problem one can follow up the factoring of the same number:

$$\begin{array}{l}
12=1\times 12\\
12=2\times 6\\
12=3\times 4\\
12=4\times 3\\
12=6\times 2\\
12=12\times 1
\end{array}$$

The amount of possibilities in the case of factoring varies with the numbers:

$13 = 1\times 13$	$14 = 1\times 14$	$15 = 1\times 15$
$13 = 13\times 1$	$14 = 2\times 7$	$15 = 3\times 5$
	$14 = 7\times 2$	$15 = 5\times 3$
	$14 = 14\times 1$	$15 = 15\times 1$

With thirteen there are only two possibilities, while with twelve there are six. The numbers 14 and 15 offer four possibilities each. The children will seek for the numbers with maximum possibilities. Naturally their amount will increase as a rule when higher numbers are reached. In order to judge which number has a relatively great amount of possibilities we can investigate how far one has to proceed from it before encountering the next one with more possibilities. One of the numbers with many factoring possibilities is 12. Any number below it offers no more than 4 possibilities, while 12 itself offers 6. Continuing from 12 the amount of possibilities is not exceeded until 24. In the range from 1 to 100 the number 60 reaches the maximum of 12 factoring possibilities, which is not exceeded by any one of the following numbers including 100 itself. This fact explains the predominating role of the numbers 12 and 60 in our units of measurement. The clock shows the number 12, the year has 12 months, the foot 12 inches, etc. An hour has 60 minutes, a minute 60 seconds, the 60th part of $\frac{1}{6}$ of the circumference of a circle equals 1°, etc.

Much that belongs to later periods of study can be prepared for at an earlier one. For instance, when taking up even and odd numbers their addition can be expressed in the following rule:

Even number + even number
= even number
Even number + odd number
= odd number
Odd number + even number
= odd number
Odd number + odd number
= even number

In later years when the subject of negative numbers is introduced a similar rule occurs:

Positive number × positive number
= positive number
positive number × negative number
= negative number
negative number × positive number
= negative number
negative number × negative number
= positive number

This apparent analogy is not accidental, but mathematically founded (even powers of a negative number are positive, odd powers negative).

When practicing long multiplication and division one may prepare the subject of roots. Instead of taking an arbitrarily chosen number for practice, one may multiply the number 1,4142136 with itself. The result is 2 followed by 7 zeros: The number 1,4142136 is the square root of 2 (up to 7 decimals), the last decimal being rounded upward so that the result of the multiplication is above 2. Then multiplying 1, 4142135 with itself the result is 1 followed by a series of nines, that is a result below 2. A similar exercise with 1,73206 and 1,73205 gives results above and below 3; by multiplying 2,23607 and 2,23606, each with itself, one result is higher and one lower than 5, etc. Multiplying $2{,}15445\times 2{,}15445\times 2{,}15445$ we get 10 followed by three zeros, etc. Then in later years when the study of roots is taken up ($\sqrt{2}=1{,}4142135\ldots$; $\sqrt{3}=1{,}73205\ldots$; $\sqrt{5}=2{,}23606\ldots$; $\sqrt[3]{10}=2{,}15444$), it will be an easy matter to proceed on this prepared foundation. But even more important than the advantages thus acquired for the progress of studies is the almost unconscious psychological effect. The pupil has the experience that his study course had since early years been built up with a deeper background than he could appreciate and this experience creates a feeling of confidence.

The way of intimating the more advanced portion of a subject in the early lessons can be extended even beyond the respective school. For instance, when practicing the four rules of arithmetic in the

early years the following scheme may be set up:

0× 0× 0=	0			
		1		
1× 1× 1=	1		6	
		7		6
2× 2× 2=	8		12	
		19		6
3× 3× 3=	27		18	
		37		6
4× 4× 4=	64		24	
		61		6
5× 5× 5=	125		30	
		91		6
6× 6× 6=	216		36	
		127		6
7× 7× 7=	343		42	
		169		6
8× 8× 8=	512		48	
		217		6
9× 9× 9=	729		54	
		271		6
10×10×10=	1000		60	
		331		6
11×11×11=	1331		66	
		397		
12×12×12=	1728			

First find the numbers which result from multiplying a number with itself three times, the cube numbers. In the next column write down the differences between the successive cube numbers, their 1st differences. In the next column are shown differences of the differences: 2nd differences. Finally follow the third differences which have the constant value 6. This fact is fundamental in differential calculus. Perhaps ten years later the student may come across this again when he will have advanced to this point in his studies.

Much can be done to arouse the interest in mathematical problems by reciprocal comparison of different kinds of numbers and methods of calculation. For instance, when introducing fractions, compare their nature with the regular numbers, not theoretically but in the form of problems. The differences between two successive fractions are not equal as they are with two successive regular numbers, but diminish continually. The step from 1 to ½ has a difference of ½; that from ½ to 1/3 a difference of 1/6, etc. In the series of numbers 1 2 3 4 5 6 7 the arithmetical mean between 2 and 6 is: $(2+6)/=4$. In the series of fractions 1, ½, 1/3, 1/4, 1/5, 1/6, 1/7 . . . the mean between ½ and 1/6 is: $(\frac{1}{2}+1/6)/2 = 1/3$. The mean 4 between 2 and 6 stands in the series of numbers in the middle between 2 and 6, whereas in the series of fractions the mean between ½ and 1/6 is not ¼ but 1/3, which stands no longer in the middle, but directly follows the first number ½. The sum of the differences between 1/3 and ¼, ¼ and 1/5, 1/5 and 1/6 is equal to that between the successive fractions ½ and 1/3.

This shows why the simple rules for addition and subtraction, applied in the case of integers, are no longer valid with fractions and that another procedure, including the finding of the common denominator, has to be used. The fractions diminish in a definite way. In order to grasp this more readily have the children hold their hands at a fixed distance from each other. This will express the unit. Then have them move the hands closer together step by step, so that the intervals gradually diminish to ½, 1/3, 1/4, 1/5, 1/6 . . . The differences here grow less and less. Now it is but a step to apply the successive fractions of lengths in the form of heights over a common base line resulting in a branch of a hyperbola. This again represents the diagram of Boyle's law in physics, in other words, the opportunity is here given to make allusion to geometry and physics which will be further developed in later school years.

What is of special importance from the psychological point of view for all school grades is the quality of thinking activity which is provoked by the methods employed. The result of arithmetic classes may be either a mechanical and primitive thinking or an intensive and creative one. There are students who when given a mathematical problem will at once ask for a fixed rule or formula with which to solve it. Their work is not based on insight.

There is no continuity in their consciousness just where the most real thought connection should set in. They usually have primitive mathematical concepts, conceiving any length as the sum of so and so many inches, any weight as the sum of so and so many grams, etc. But neither a length nor a weight is composed of parts, each forms a whole that is to be compared with others. A more intensive thinking will be more apt to aim at comparisons, at proportional numbers and less at dissecting and combining. If a measure is three times that of another, this fact remains whether the measuring be done in meters or inches. Nearly all the numbers used in the sciences are fundamentally proportional numbers. If a stone is said to have specific gravity 3, that means that its weight is three times that of an equal volume of water; it is not a combination of three units.

Psychological effects produced by various kinds of mental activities are naturally not limited to their respective realms. They continue to act in the conscious as well as in the subconscious spheres of our mind and their influence finally affects even social life. No healthy relationship and cooperation between human beings can be founded on mechanical and one-sided thinking. Such thinking isolates one within oneself and builds no link toward the understanding of the other person's point of view. Thus it is often the cause of dissatisfaction and of a lack of self-confidence. It leads to an attitude of self-defense toward life without courage or initiative, or to a uselessly critical and cynical attitude.

The psychological effect of creative studies in arithmetic is at least of equal importance as the knowledge the subject imparts. Whoever looks only for the "practical" side in arithmetic readily overlooks the truly practical effects within the psychological realm. Particularly in our days this becomes increasingly essential. The dangers threatening culture and civilization today are not due primarily to any lack of detailed practical knowledge, but rather to psychological factors which, in the final analysis, are the outgrowth of distorted thinking. The teaching of arithmetic in its education of sound and straight and creative thinking holds a key position in our work for a future society throughout the world.

The Mathematics Teacher wishes all of its readers a Merry Christmas and a Happy New Year!

The Teaching Objectives in a First Course in the Calculus

By James E. Parker
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There is a great deal being said nowadays about course objectives. It is said that the objectives for any course in the curriculum should be clearly defined and that teacher and student alike should be conscious of these objectives at all times. This concern has lead to a re-examination of the objectives for courses taught by the writer. This paper is particularly concerned with the objectives for a first course in differential and integral calculus. Stated briefly, the question at hand is this *what are my objectives in teaching a first course in the calculus?* Three major objectives are found to be inherent in this case. These are: (1) to give the student an understanding of the fundamental concepts of the calculus and a point of view relative to the historical background out of which these concepts grew; (2) to develop proficiency in the manipulative skills of differentiation and integration; and (3) to develop abilities in making practical applications of the principles learned. A few words will be said about each of these objectives relative to the methods used in attempting to realize them. It is hoped that the following statements will arouse some interest and will bring forth some comments as well as criticisms from individuals more experienced than the writer. In this way the writer, and possibly others, may be greatly helped in setting up goals toward which to work.

In the first place, we are assuming that a knowledge of the fundamental concepts of the calculus is essential to a successful effort to the study of the calculus. The concepts of a function, a variable, increment, limit, and continuity are absolutely necessary stepping-stones for the beginning student, and the well-equipped teacher "is prepared to put into proper perspective before his students the most fruitful concepts of his field."[1] These are the words of Dr. C. E. Van Horn, who says further:

> In introducing students to new concepts, nothing can be safely taken for granted. The instructor must know that the students are clear on all the little points that surround the subject.
>
> An instructor must take time to encourage habits of clear thinking. And nothing, perhaps, is more conducive to the cultivation of such habits than to be clear-cut in characterizing logically the concepts with which one may be dealing.

It is indeed difficult to imagine a student making strides of progress if he is lacking in a clear knowledge of such a concept as the limit, for example. It seems that first and foremost in the minds of calculus teachers must be a definite aim to teach the underlying concepts of the calculus. This should not be done as a separate unit of work but the concepts should be introduced as the need for each arises.

One method of realizing this objective is to have the student write brief papers in which he is required to discuss certain important concepts which are being studied at the time. The student should be encouraged to make crystal clear the concepts discussed. Through complete statements and through a strict choice of words the student soon develops the ability to give real meaning to these concepts. Such a paper enables the teacher to locate discrepancies in the student's understanding, and, when properly handled by the teacher, reveals to the student his own errors in thought, thus allowing for self-evaluation in growth.

Secondly, we must always consider the

[1] Van Horn, *A Preface to Mathematics*, Chapman and Grimes, Inc., 1938.

skills accompanying the study of practically every elementary course in mathematics. A successful calculus student is certainly not remote from these skills. Calculus can hardly be taught without some consideration of a degree of proficiency in the skills of differentiation and of integration. We should, however, take care that we do not over-emphasize this aspect of teaching. Too often courses in elementary mathematics turn out to be no more than a routine process of *stereotyped problem solving*. Apparently the better way to approach this phase of the work is to work on a selective basis. That is to say, select problems which best suit the needs of the hour and best illustrate the principles at hand. This is a more flexible method than that of merely following the text through problem by problem. This method may also sidetrack any tendency for teacher and student to become textbook slaves. There is, on the other hand, a danger of underemphasis of the development of the basic skills. An over-zealous teacher is likely to become bored by what may seem to him drill work. Most calculus textbooks have a large number of problems. This teacher is likely to feel that too much emphasis is placed thereon. He should realize that a student can hardly find his calculus at his command without proficiency in these basic skills. The ability to make practical applications of calculus is in a very large measure dependent upon these skills. For after all, practical application of any phase of the calculus is a process of finding a satisfactory answer to some problem. This leads to our third objective: namely, to develop abilities to make practical applications of the principles learned.

For the past quarter of a century or more we have witnessed a decline in the teaching of mathematics in our secondary schools. Educators have been prone to believe that little mathematics is needed by the average man. Consequently, what has actually happened is that the selective few whom they consider in need of the subject have suffered in their early work. This is particularly true of the small schools. If on the one hand, mathematicians have been prone to over-emphasize pure mathematics, educators have, on the other hand, failed to recognize the value of pure mathematics in its service capacity, its advisory capacity. We are here considering the phrase *practical applications* in a much broader sense than the mere use of the subject as a tool for engineers. If we examine the records we will doubtless find that the work of mathematicians has very often been a reservoir for many hints or helps along practical lines. For a thorough discussion of this aspect of the teaching I refer you to Mr. Thornton C. Fry's article "Industrial Mathematics," *The American Mathematical Monthly*, vol. 48, June–July 1941, Part II, Supplement. Here are listed some fine examples of the very thing of which we are speaking. These are supported by the best of evidence. Indeed all mathematics is practical in several ways. To be sure the word practical infers usefulness. But a thing may be useful in a number of ways.

Teachers of calculus should be alert in attempting to find ways of bringing home to the student the various applications of his subject. This can be done in seemingly insignificant ways. For example, in working the following problem in maxima and minima a student of farm experience found that in a small way his calculus could help him out on the farm:

> An open trough is to be made from a long rectangular shaped piece of metal by bending up the long edges so as to give the trough a rectangular cross section. If the width of the piece is 1 foot, how deep should the trough be made in order that its carrying capacity may be a maximum?[3]

This may seem rather trivial, but the point is this. This young man saw the usefulness of his calculus and subsequently was motivated to study calculus more conscientiously. Students as far along as this may

[3] Ford, W. B., *A First Course in the Differential and Integral Calculus*, Revised edition, Henry Holt and Company, New York, 1937.

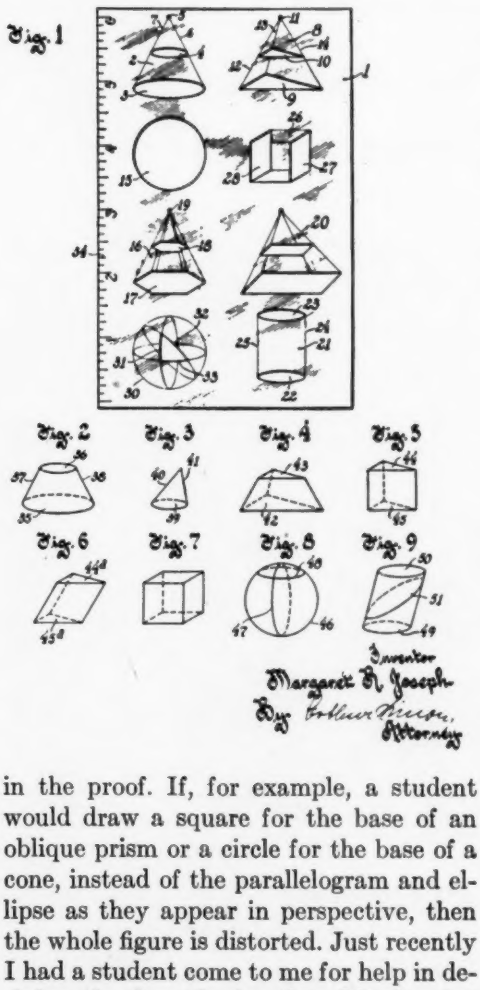
often need to be motivated in this way. As a matter of fact, college students are more and more demanding insights into the usefulness of their subjects, and the calculus teacher has an unique opportunity in this connection.

The writer is, of course, teaching in a liberal arts college, but he feels that any student taking the course prescribed for above should find himself ready and capable in whatsoever field of endeavor he may find himself. The statements made herein are merely suggestive. No attempt has been made to write an extensive discourse on the topic. The purpose is to bring to a focus the objectives set forth by the writer.

Accurate Drawings of Three Dimensional Figures*

By Margaret Joseph
Milwaukee, Wisconsin

Every teacher of solid geometry has no doubt experienced having a student fail to prove an original exercise just because his drawing of the figure required was so much out of proportion that he failed to establish the proper relationship between lines, angles, triangles, or planes necessary



in the proof. If, for example, a student would draw a square for the base of an oblique prism or a circle for the base of a cone, instead of the parallelogram and ellipse as they appear in perspective, then the whole figure is distorted. Just recently I had a student come to me for help in deriving the formula for the altitude of a regular tetrahedron in terms of its edge *e*. As soon as I saw the rough sketch he was using, where the faces were scalene triangles, I knew why he had two unknowns in the equation that he was trying to solve. The altitude did not meet the base of the figure at its center. I made a better drawing for him and he immediately recognized his error and proceeded with the solution without further help from me.

On the other hand there are those students who have had training in isometric or perspective drawing and they often do such meticulous work that they spend hours on the drawings and solutions when several original exercises are assigned for homework. This may create a serious situation. The student spends much more time on his geometry than he can afford to so either he will in turn neglect his other subjects or else he may develop a dislike for the subject just because it demands so much of his time.

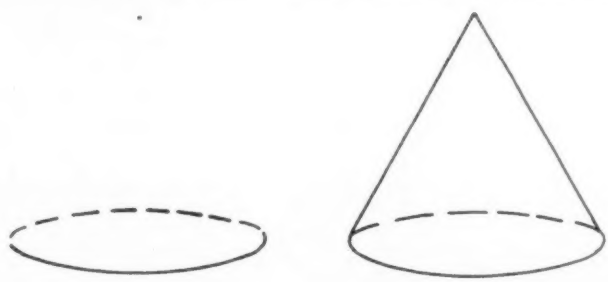
Models of solids made professionally or by students provide an excellent means of clearing up any misunderstanding the students have in regard to the figure. Those made of wire are especially fine to use in visualizing intersecting planes or surfaces. However, they do not help the student lessen his labor in making drawings for theorems or original exercises assigned for homework.

To overcome this difficulty the student may use a templet or stencil to facilitate the drawing of three dimensional figures studied in solid geometry. It consists of a sheet of transparent material such as celluloid or plastic and is provided with a number of cutouts to represent isometric projections of solid figures. This templet will enable the student to make an accurate drawing of the desired solid figure in just a few seconds' time.

Figure 1 in the drawing shows the templet with the isometric cutouts shown in heavy lines while the lighter lines associated with them form the completed solid

figure. Thus the student can see at a glance some of its most common uses. Figures 2 through 9 show a few of the other solid figures which can be drawn with this templet but with a little practice the student can draw others by combining cutouts of two different figures.

The cone in Figure 1 shows the isometric projection of the base and of a plane parallel to the base. Those sections or cutouts are numbered 3 and 4. The tiny hole at the top, numbered 5, fixes the vertex of the cone. If a right circular cone is desired, outline the elliptical section numbered 3 with the point of a well sharpened pencil. (With a little practice the student will know just which lines or parts of lines to have dotted and which should be solid. Until he masters the technique however, all lines can be made solid and then use an eraser to make dotted those which should be.) Then to locate or fix the vertex, insert the pencil point in the opening numbered 5. Now use the edge of the templet as a straightedge or ruler to draw lines from the vertex to the extremities of the base as shown in the drawings above.



If a smaller cone is preferred, use the section parallel to the base of the cone, numbered 4 on the drawing instead of the one numbered 3, and proceed as before. If an exercise calls for an oblique cone, outline the base just as before. Then move the templet right or left before fixing the vertex and you will have a drawing similar to that shown in Figure 3. In each pyramid shown on the templet as well as in the cone, the hole which fixes the vertex is directly over the center of the base of the figure. That is why the templet must be moved right or left after outlining the section which forms the base, if the figure being drawn is to be oblique. Figure 2 shows a frustum of a right circular cone which is drawn by outlining the section of the cone parallel to the base and the base. Then the edge of the templet is used to draw the two slant heights necessary to complete the figure.

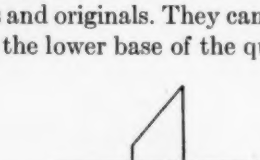
The pyramid shown at the top of the second column in Figure 1 can be used to draw several figures. Just as the cone and its frustum were drawn, one can readily see how a pyramid can be drawn with cutout numbered 9 as base and the hole numbered 11 as its vertex or a smaller pyramid drawn with cutout numbered 10 used as base instead of 9. Similarly a frustum of a triangular pyramid can be drawn with cutouts 10 and 9 as bases. To draw a triangular prism, outline cutout numbered 10 and then move the templet directly up or down on your paper and again outline the same cutout to form the two bases of the prism. Then use the edge of the templet to draw the lateral edges of the prism as shown by the completed drawing in Figure 5. If cutout numbered 9 is used instead of 10, then a prism with larger bases would result. Even if the smaller bases are used to draw a figure, a larger solid than that shown on the templet can be drawn by increasing the altitude. This is done by moving the templet up, after outlining the base of the figure being drawn, and then fixing the upper base if the figure is a prism or its vertex if the figure is a cone. If auxiliary lines need to be drawn or if nu-

merical values are put right on the drawing, it is wise to use a larger figure.

Just as the prism was drawn using either cutouts 9 or 10 as bases, one can draw cylinders using as bases cutouts numbered 3 or 4, the isometric projection of the base of the cone and the plane parallel to it. However, the cylinder shown in the lower right hand corner of the templet can generally be used. If a cylinder with a larger or a smaller base is preferred or if one wishes to draw 2 similar cylinders, then use cutouts 3 and 4

This templet also shows pyramids with bases of 4 and 6 sides to its base. These too can be used to draw just plain pyramids, their frustums, or right and oblique prisms with quadrangular and hexangular bases. The manufactured templet will also show one with a pentagonal base as well as a 30, 45, and 60 degree angle. These same pyramids can be used to draw other polyhedrons. For example, to draw a regular hexahedron, first draw a triangular pyramid in the usual manner. Now turn the templet over and let the longest edge of the base of the pyramid on the templet coincide with the corresponding edge of the pyramid just drawn and insert the pencil point in the hole which is numbered 11. This vertex is below the pyramid instead of above. Use the edge of the templet to complete the remaining lateral edges of the hexahedron. In like manner the other pyramids shown can be used to draw polyhedrons of 8 and 12 faces.

Parallel planes are used in a number of theorems and originals. They can be drawn by using the lower base of the quadrangu-



lar pyramid for one plane and then if the templet is moved up or down the desired distance, the same cutout should be outlined for the plane parallel to the first. Two perpendicular planes are used in a whole family of theorems. The base of the quadrangular pyramid should be used for the horizontal plane and cutout numbered 27 or 28 for the vertical plane as just shown.

In the study of spheres, one frequently needs to draw a meridian or a parallel of latitude or show sections of a sphere made by a small and a great circle. The circular cutout numbered 13 should be outlined to represent the earth or any sphere. Cutout numbered 3, shown as the base of the cone, will just fit for the equator when in a horizontal position and as a meridian or any other great circle when in a vertical or oblique position. Cutouts numbered 22 and 23 can be used as small circles of the earth or as other parallels of latitude. Figure 8 shows cutout numbered 3 as a meridian and cutout numbered 23 as a small circle or section of the sphere. In this figure they are numbered 47 and 48 respectively. If the students would want to show a lune of a sphere, place cutout numbered 13 in position as if drawing a meridian but instead of drawing half with a solid line and half with a dotted line, draw the entire cutout with a solid line and whole figure will appear to be on the same side of the sphere. Spherical triangles are always difficult for the student to draw and usually they appear as plane triangles on their drawings. Therefore the figure at the lower left of the templet has been included. Here the intersecting arcs of the great circles of the sphere are represented by the cutout formed by arcs which are numbered 31, 32, and 33. The circular cutout numbered 15 is identical in size to the sphere represented by the circle numbered 30. So if a spherical triangle is to be drawn, first outline the circular cutout numbered 15. Then place the circle numbered 30 so it coincides with it and outline the cutout formed by arcs numbered 31, 32, and 33.

The commercially manufactured models of solids or those prepared by students are not always satisfactory for the student to use to demonstrate a theorem in the

classroom. I have used them successfully in some cases where the proof of the theorem was very short and easy to follow and when the class is small enough so that each one can see the model clearly. If the theorem being proved is at all involved, one must have a drawing with the faces lettered so that the student can follow through the proof presented. There are times when parts of the proof must be written on the blackboard. This is true for example, in the derivation or proof of the volume of a regular pyramid. ($V=\frac{1}{3}h\ [B+b+\sqrt{Bb}]$). It is true of other theorems where there is a long logical sequence of steps either geometric or algebraic in nature. For such cases or for the figure needed when original exercises are explained at the blackboard, a good drawing is essential. I have seen students spend 10 to 15 minutes of class time to put the drawing of the solid figure needed for a theorem and still the class had to stretch their imaginations to recognize the true figure. I once visited a class in solid geometry where a student spent so much time putting a drawing for the theorem on the board that there was not enough time left to finish the proof. Templets for blackboard use would eliminate this problem. One single templet like that described above with all figures on would be too cumbersome to use and certainly to house if large enough for blackboard use. Hence several different templets can be used each of which has but one figure similar to that described in the single templet only much larger in scale. They could be housed in the average teacher-desk drawer or in the pasteboard carton in which they are shipped.

If all teachers of solid geometry could use the blackboard templets, more class time would be available for the discussion of work assigned. Class time would not be wasted in making drawings of solid figures and hence more time would be left to present the advanced assignment to the student. The student would then have more time to ask questions about the lesson before he leaves the classroom. Furthermore he would grasp the presentation of the proofs more readily if accurate isometric drawings are used. Then when he uses the pocket size templet to make the drawings required in preparation of homework, the time spent would be reduced to a minimum and the most ideal situation for a solid geometry class would result.

(Reference to numbered figures is to the drawings shown p. 350.)

To Autumn

Season of mists and mellow fruitfulness,
Close bosom-friend of the maturing sun;
Conspiring with him how to load and bless
With fruit the vines that round the thatch-eaves run;
To bend with apples the mossed cottage-trees,
And fill all fruit with ripeness to the core
To swell the gourd and plump the hazel shells
With a sweet kernel; so set budding more
And still more, later flowers for the bees,
Until they think warm days will never cease,
For summer has o'erbrimmed their clammy cells.

Who hath not seen thee oft amid thy store?
Sometimes whoever seeks abroad may find
Thee sitting careless on a granary floor.
Thy hair soft-lifted by the winnowing wind;
Or on a half-reaped furrow sound asleep,
Drowsed with the fume of poppies, while thy hook
Spares the next swath and all its twined flowers:
And sometimes like a gleaner thou dost keep
Steady thy laden head across a brook;
Or by a cider-press, with patient look
Thou watchest the last oozings hours by hours.

Where are the songs of Spring? Ay, where are they?
Think not of them, thou hast thy music too—
While barred clouds bloom the softly-dying day,
And touch the stubble-plains with rosy hue;
Then in a wailful choir the small gnats mourn
Among the river sallows, borne aloft
Or sinking as the light wind lives or dies;
And full-grown lambs loud bleat from hilly bourne;
Hedge-crickets sing; and now with treble soft
Redbreasts whistle from a garden croft;
And gathering swallows twitter in the skies.
In the gay woods and in the golden air,
Like to a good old age released from care,
Journeying, in long serenity, away,
In such a bright, late quiet, would that I
Might wear out like like thee, 'mid bowers and brooks,
And, dearer yet, the sunshine of kind looks,
And music of kind voices ever nigh;
And when my last sand twinkles in the glass,
Pass silently from me, as thou dost pass.

—*William Cullen Bryant.*

Fun for the Mathematics Club

By Jacob M. Porterfield
State Teachers College, Maryville, Missouri

With the teachers carrying a war-time load, the mathematics club is apt to die for lack of anything interesting to do. There just isn't sufficient time for the instructor to make the necessary preparations to keep everything going. The literature intended to be helpful is often so general that its value is lost. Some of the literature appears in the nature of a preachment. This wastes almost as much time as the articles which quibble over "objectives." To give them something specific to do, it is suggested that mathematics clubs, as a part of their activity, build an interesting and useful calculator which will acquaint them with much elementary algebra and trigonometry. It does not require logarithmic scales and can be drawn with an ordinary straight edge and compass with a satisfactory degree of accuracy.

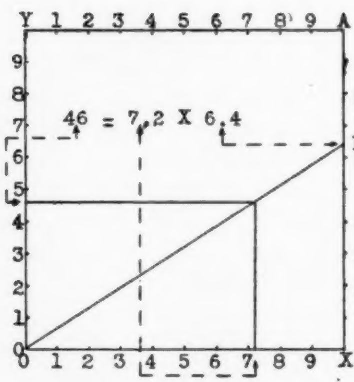


Figure I

Cardboard, graph paper, cellulose tape, a strip of transparent plastic, and a small rivet or ledger screw are all that are required. As in Figure I, a square of graph paper, ten units on a side, is mounted on cardboard with cellulose tape placed along its edges. A thin straight line is etched and inked from end to end on the plastic strip. A scale equal to one dimension of the graph is traced along this line. Rectangular axes are drawn on the graph paper so that it represents the first quadrant. The plastic strip is riveted so that the zero point of its scale is exactly at the origin and is free to turn about that point. Circles, lines and scales are drawn on the square in a manner that will be made clear as we proceed.

Suppose it is desired to multiply any number by 6.4. Consider Figure I. The line OP on the plastic strip is turned until it intersects XA at 6.4. To find the product of 6.4 and 7.2, find 7.2 along OX. Follow upward to the intersection of OP and to the left to read the product along OY. The answer is 46. The decimal point is located in the same manner as with a slide rule. Any other number can be multiplied by 6.4 in the same manner. The solution is general.

To get the same accuracy for numbers beginning with the digits 999 to 156, it is necessary to make a new setting of OP. Disregarding the decimal point, OP cuts the lines $X=10$ and $X=1$ at the same value of Y. Rotate OP counterclockwise until it passes through the point (1; 6.4). The product of numbers between 999 and 156 times 6.4 can now be read exactly as before. It should be noted that 64 and 156 are reciprocals, if the decimals are disregarded.

The process of division is obvious. To divide several numbers by 6.4, the line OP is set as in Figure I. To divide 46 by 6.4, find 46 along OY, follow to the right to the intersection of OP and down to read the quotient, 7.2, along OX. Any number from 000 to 6.4 can be divided by 6.4 without resetting. The numbers from 640 to 999 can be divided by 6.4 if we again remember that OP cuts the lines $X=10$ and $X=1$ with the same sequence of digits.

Rotate OP counterclockwise until it passes through the point (1; 6.4). The division is preformed as before.

In a first consideration, it would seem that the square would need to be made larger as OP is rotated more and more in a counterclockwise direction. To further illustrate that this is not true, Figure II should be considered. Here 16 is in the multiplying position.

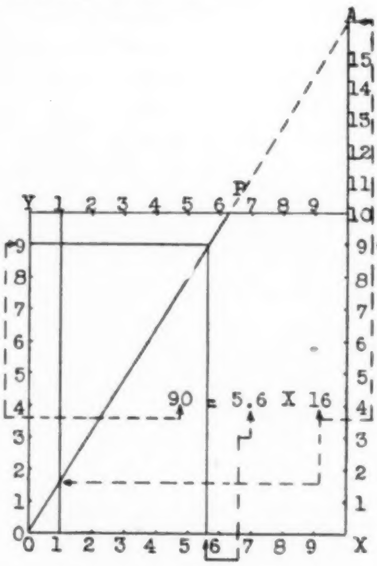


FIGURE II

It seems necessary to extend the $X=10$ line to A. But the same results are obtained by setting OP to pass through the point (1; 1.6) along the line $X=1$. To multiply 5.6 by 16, find 5.6 along OX, follow upward to the intersection of OP and to the left to read the product, 90, along OY.

Any number starting with the digits 160 to 999 can be divided by 16 without resetting. Find 90 along OY, follow to the right to the intersection of OP and down to read the quotient, 5.6, along OX. To divide numbers starting with the digits 999 to 160 by 16, it is necessary to rotate OP clockwise until it passes through the point (10, 1.6) and make the readings as before.

Reciprocals are found by dividing the number into one. Consider Figure III. To find the reciprocal of 28, set OP to pass through the point (2.8; 1.) Find the point where OP intersects XA and read the Y value. Locate the decimal point. The answer is .036.

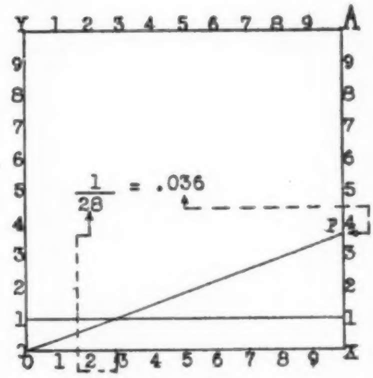


FIGURE III

It is possible to first locate the number along OX. To find the reciprocal of 3.6, turn OP so as to pass through the point (10; 3.6). Find the point where OP intersects the line $Y=1$. and read the X value. The answer is .28.

This relationship could be considered from the trigonometric point of view. The tangent is the reciprocal of the cotangent. The methods of finding reciprocals from the trigonometric relationships can be worked out by the club.

Another method of multiplying and dividing is worth considering. In Figure IV a circle has been drawn about the origin

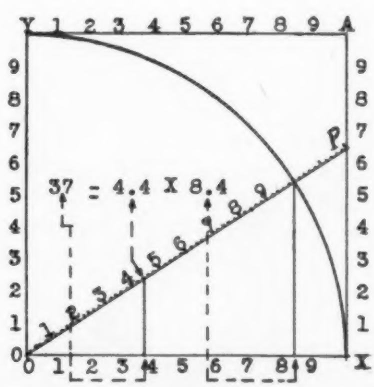


FIGURE IV

with a radius of ten units. To multiply any number by 8.4, set OP so that it passes through the point where the line, $X=8.4$, intersects the circle. As an example find 4.4 along OP and follow down to read the product $4.4\times 8.4=37.$ along OX. All other numbers can be multiplied by 8.4 in the same manner without resetting. To read the product of 1.1 times 8.4, greater accuracy is obtained by finding 11. on OP and reading the X value along OX. The product is 9.2.

To divide 8.4 into any number set OP at the intersection of the line, $X=8.4$ with

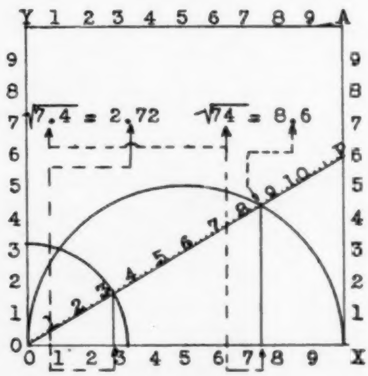


Figure V

the circle. Read the dividend along OX and the quotient straight above on OP. For example; 37 divided by 8.4 is 4.4 and 92 divided by 8.4 is 11. In this last example find 9.2 along OX. Follow straight up to the line OP. Again the solution is general.

The above method of dividing is good if the divisor starts with a larger digit like 7, 8, or 9. It runs into difficulty as OP is rotated more and more in a counterclockwise direction. Another consideration helps us to overcome this difficulty. Consider again Figure IV. To divide 37 by 4.4 rotate OP until 4.4 on OP is directly above the value of 3.7 on OX. Find the place OP intersects the circle and follow down to read the quotient, 8.4, on OX. The reciprocal 4.4 divided by 37 can be read on the scale of OP at the intersection of XA.

In order to find squares and square roots it is necessary to draw two more circles as in Figure V. With a radius of five units, a circle is drawn with the point (5; 0) as a center. With a radius of $\sqrt{10}$ units, a circle is drawn with the origin as a center. In the latter case to get the exact radius set one point of the compass at the origin while the other is at the point (3; 1).

To find the square root of 74, consider Figure V. Find 7.4 along OX. Follow straight up to the intersection of the large circle. Turn OP so as to pass through this point. Read the square root, 8.6, on the scale of OP.

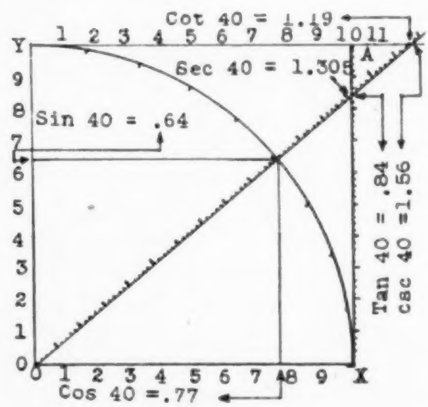


Figure VI

To find the square root of 7.4, make the same setting as before. But to read the answer, find the point where OP intersects the smaller circle, follow down to find the square root, 2.7, on OX.

For any number with an even number of digits to the left of the decimal point, use the first process. For any number with an odd number of digits to the left of the decimal point use the last process. Moving the decimal point two places in the number is equivalent to moving it one place in the answer.

The solution of the right triangle is simple. It is possible to turn OP to the desired angle and not only read any of the six trigonometric functions, but also multiply or divide their value by any number without resetting OP.

Suppose it is desired to solve $M = 9.3$ Tan 40°. OP is turned so the angle XOP is 40°. A protracter scale should be marked off along the circle, for this purpose. The tangent of 40° is read along XA. It is .84. Remembering the method of multiplying, used in Figure I, find 9.3 along OX, follow up to OP and to the left to read the value of M (which is 7.8) along OY. Any other number can be multiplied times the tangent of 40° without resetting it. The solution is general.

The other trigonometric functions can be read after locating the decimal point. The cosine is read along OX; Cos $40° = .77$. The sine is read along OY; sin $40° = .64$. The cotangent is read along YA; Cot $40° = 1.19$. The secant is read along OP; Sec $40° = 1.305$. The cosecant is read along OP; Csc $40° = 1.56$. It must not be forgotten that the same readings may be made along $X = 1.$ as along $X = 10$, and likewise the same readings may be made along $Y = 1.$ as along $Y = 10$. For this reason it is not necessary to make the square larger.

Students will find it interesting to devise calculators for special uses. Adjusting the decimal to a suitable range and labeling the scales enables the student to become proficient in the use of the calculator with great ease. Permitting students to adapt their device to their own special interests will add zest to their work.

For example, in Figure VII, the relationship of voltage, current, and resistance according to Ohm's law is shown for a special range of resistors.

To find the various voltage drops across a 5,000 ohm resistor for various current values flowing through the resistor, turn OP to 5,000 ohms. 4.4 milliamperes will flow when the drop is 22 volts. Other values may be read without resetting. Moving the decimal one place to the left in the ohm scale, moves the decimal one place left in the volt scale. If it is necessary to read ohms along the line $X = 1.$ the decimal must be moved one place to the right.

Such devices can be made of suitable range for any Ohm-law problems.

Physics students may have fun building a calculator for the composition or resolution of forces. Practical radio students will be interested in the solution of the impedance triangle. Students of aeronautics

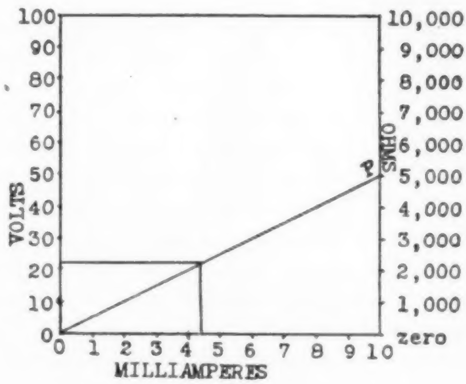


Figure VII

can make many applications to the solution of their problems. Working out proofs for the various applications should keep the best students busy for a long time with work that is worthwhile.

It is not uncommon for a club to begin with a variety of popularity contests wherein students are elected to offices with no real duties to perform. The names are published in the school paper and the principal is happy that the school has a new club. Activities often consist of framing a constitution and time is wasted in reading the minutes of the last meeting. To avoid such nonsense, students must have something definite and worthwhile to do.

The CAA-NYC Aviation Operations Institute

By Lillian Moore
Far Rockaway High School, New York, New York

The Civil Aeronautics Administration —New York City Aviation Operations Institute was held on April 6th at La Guardia Field, New York City. It was arranged by Mr. O. P. Harwood, Regional Manager of the First Region of the Civil Aeronautics Administration for ninety-eight selected New York City teachers of aeronautics and school administrators, in conjunction with Dr. Frederic Ernst, Associate Superintendent in Charge of High Schools. The Institute program proved to be valuable to the highest possible degree. Teachers who were unable to take advantage of the opportunity offered to gain increased knowledge of aviation operations will undoubtedly be interested in a report of the activities covered during the course of a busy day.

After registration and group assignment, I attended a series of lectures given by members of the CAA personnel on airport and airways engineering, airways and airport traffic control, radio and teletype communications, air carrier inspection, accident investigation, and aviation training programs. Visual aids were used effectively in emphasizing key points in well organized lectures. Civil Aeronautics Board and Civil Aeronautics Administration organization, duties, and investigations were explained. An air carrier inspector discussed instrument approaches, minimum ceiling and visibility for takeoffs and landings, lighting facilities, and auxiliary airports. There are three types of CAA inspection: operations, maintenance, and radio. A Civil Aeronautics Board lecturer explained the organization of the CAB. He discussed the accident investigation operations in promoting safety, the use of specialists in aircraft, engines, and meteorology in the analysis of accident reports, and the use of field men as investigators. The functions of the Civil Aeronautics Board in regulation, rule making, and education were discussed. In a talk on the aviation training program the lecturer reviewed the history of civilian pilot training, war training service, the development of courses, and the introduction of visual aids in flight training.

Another member of the CAA personnel described the in-service training program established for new employees. Control tower operators are trained in the air traffic control school to use radio and light signals. They practise controlling takeoffs and landings at a local airport and following flights along an airway or route traffic control board. In the communications school radio operators are trained in Morse code and teletype transmission and receiving. In the signals school a course in radio theory is given. This course includes the repair of radio equipment along airways, teletype machines, and radio range stations. An airport engineer discussed airport management, runways inspection, and the use of taxiways and turf areas between runways. An airways engineer explained the planning of airways routes, location of emergency fields, installation of beacons, voice broadcasts of weather, identification of high frequency fan and zone markers, and the operation of an instrument landing system. The chief of the general inspection branch discussed the monthly itinerary of inspectors, pilot ratings, ground instructor, flight instructor, and mechanic examinations, and written, practical and flight tests.

Weather bureau personnel described the operations of this division in the use of recording instruments, method of taking weather observations, the collection, compilation, and analysis of these data, and their use in developing charts for weather forecasting. Weather stations along the airways make hourly surface observations,

visual and instrument. Ceiling, temperature, dew-point temperature, wind direction and velocity, and barometric pressure are measured. During the day helium balloons with a known rate of ascension are used to measure the ceiling. At night projector lights sending vertical beams are used for this purpose. Psychrometers, wind vanes, cup anemometers measuring wind velocity, and barographs measuring barometric pressure were observed in operation at the weather bureau station. Upper atmosphere observations were explained. Pibal observations obtained from pilot balloons followed with a theodolite to measure the direction and intensity of winds at 1000 feet elevations, and the radiosonde which measures temperature, humidity, and air pressure were exhibited.

The Transcontinental and Western Air hangar was visited, so that airplane repairs and maintenance could be observed. Their Link trainer operator explained the use of that instrument in checking pilot reactions. CAA pilots gave every visitor a thirty-minute trip in a Cessna twin-engine plane. Flying at a 3500 foot altitude and speed of 135 miles per hour, the plane covered a considerable area around the field. The air was somewhat bumpy over Long Island. Earphones enabled the passenger to follow the pilot, control tower operator conversations, illustrating local airport control. Visits to the communications operations centers, illustrating the operation of teletype circuit wires, radio telegraph and telephone communications, radio direction finder station, and trans-Atlantic communications to and from planes were particularly instructive. The TWA traffic control board was viewed. One of the high points of the Institute was the visit to the airport control tower, where the operator was observed guiding planes in their takeoffs and landings at the field. Lunch was served with the compliments of Eastern Airlines, Inc. An open forum after dinner in the Terrace Room concluded the activities of a day well spent. A lively discussion on aviation education ensued during the forum.

I am certain that the information which I obtained as a result of following the program of activities will prove to be most helpful and instructive, when I relay it to the members of the aeronautics classes. The CAA should be complimented on the well planned, perfectly organized program, and on the efficient manner in which it was carried out. Mr. Harwood deserves the utmost praise for instituting this means of advancing in-service aviation education.

Virtus ex Necessitate

By MABEL POUND ADAMS, *Sacramento, California*

Space, rhythm, beauty:
These come to me out of the pages
Of my worn algebra book.
And sometimes, when I'm sleeping,
Transparent forms glide near—
Then Galileo, Newton, Napier,
Flick ghostly fingers against my eyelids,
Saying, "O foolish one, have you come to us so late?
Out of a mass of nothings, do you at last remember
Immutability through orderly change?J
Can you hear the stars singing in their courses
In a universe deep and full of swift motion?
Do you sense at last
That space cannot be measured until it is bounded?"
Then, perhaps, dreaming, I stir. In the morning
Value and quantity have taken on deeper meanings.
Then the relentless truth of an equality,
The everlastingness of an identity,
The inescapable dependence
Of one thing upon another—
These are harmonious, and fit together
In the balanced patterns
That come out of the pages of my dog-eared algebra book.

Continuity in Geometry

By Philip Peak
Indiana University, Bloomington, Indiana

The dictionary defines continuity as "connection uninterrupted." It certainly would be too much to expect a course in geometry to be a connection uninterrupted. However when the interruptions occur almost daily we cannot say we have even a degree of continuity. True, the daily work uses for basis of proof or substantiating reasons, those facts previously learned, but it seldom aids the pupil in seeing the development of one idea into a more general one. Or of deducing from the general idea a more specific one.

By continuity in geometry we mean, that relationship of one theorem to the next, what things do they have in common, or better still what is the one factor which causes the second to be different from the first. This implies that all other factors are the same. It is not maintained that an entire course should be linked so closely together, but certainly there are many sets of problems and theorems where a single change has produced a new problem. It is a well established fact that the learning process is proceeding from the known to the unknown. We base our assumptions on those known facts which we interpret in the light of our past experiences. Since this is the case teaching plane geometry may be made not only easier but more effective by emphasizing continuity. Very thorough analysis is sometimes needed to bring out relationships which show continued development. However the time devoted to such activity is well spent and both teacher and pupil are doubly repaid for their efforts.

The purpose of this paper is to list a number of theorems found in the average plane geometry book and show their continuity. In many instances it is impossible to teach such theorems during the same short period of time. This fact does not alter the value of continuity. In some cases time increases the value of it as a teaching aid. The solution of original exercises is an example of continuity, if we analyze the problem in the light of those facts on which it is based. There will be no attempt to include problems in this paper because they vary a great deal and the process is similar to that of theorems and corollaries.

Suppose we take for the first example the congruency theorems:

1. *If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the two triangles are congruent.*

2. *If two angles and the included side of one triangle are equal respectively to two angles and the included side of another triangle, the triangles are congruent.*

3. *If three sides of one triangle are equal respectively to three sides of another triangle, the triangles are congruent.*

Here we have no continuity since either one is a complete fact or postulate in itself. But if we look at the fourth congruency theorem namely: "two right triangles are congruent if the hypotenuse and adjacent angle of one equals respectively the hypotenuse and adjacent angle of the other"; we find it to be a special case of number two. The conventional proof makes no mention of this since the student has not proved there are 180 degrees in all the interior angles of a triangle. To prove that the sum of the angles of a triangle equals 180 degrees we need parallel lines therefore the above theorem could well be postponed until that stage is reached. In the same manner we could show the following theorems:

1. *Two right triangles are congruent if a leg and an adjacent acute angle of one equals respectively a leg and an adjacent acute angle of the other.*

2. *Two triangles are congruent if two*

angles and any side of one are equal respectively to two angles and any side of the other.

The last congruency theorem namely, "Two right triangles are congruent if the hypotenuse and side of one are equal respectively to the hypotenuse and side of the other"; does not follow the pattern but is a special case of the fact that every point on the perpendicular bisector of a line is equidistant from the ends of the line. When studying the proposition this point should be pointed out regardless of which fact is studied first.

The continuity involved in the quadrilateral problems is fairly obvious. Beginning with the general quadrilateral and making changes one step at a time we can follow through the entire field as found in plane geometry texts. There will be some variation as to the definition of a parallelogram but the starting point makes very little difference if we base our further proof on the beginning. The general quadrilateral is a closed figure composed of four line segments. Change one side of it so that side is parallel to another and we have the trapezoid. Change one side of the trapezoid so opposite sides are parallel and we have the parallelogram. Change one angle of the parallelogram to a right angle and we have the rectangle. Likewise the square and rhombus may be gotten from the parallelogram and rectangle. The so-called parallelogram theorems can be developed in order beginning with, "Either diagonal of a parallelogram divides it into two congruent triangles." As each problem is developed the special condition of a rectangle, rhombus and square can be added to show how more conditions on the original figure gives a more specific picture of the result. A similar program can be followed with the converse of these theorems.

One of the best known examples of continuity is found in circles and angles measured by their arcs. The fundamental theorem being, "An angle formed by two lines cutting a circle is measured by $\frac{1}{2}$ the sum of the arcs intercepted." This includes all cases; central angles formed by two radii, inscribed angles formed by two chords, angles formed by a tangent and a chord, and angles formed by two chords. When we use the angle formed by two secants or a secant and a tangent the same theorem holds but positive and negative arcs must be used. This set also includes the theorem, "Parallel lines intercept equal arcs on a circle," because the angle between the lines is zero and the algebraic sum of the arcs is also zero.

Similar triangles give us a good example of growth. By going back to congruent triangles we can set up similar triangles with less restrictions. From this beginning we go through the Pythagorean theorem, Hero's formula for the area of a triangle, and on to the law of cosines if you wish to carry it that far. Using similar triangles to prove, "The altitude to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse" and "either leg is the mean proportional between the whole hypotenuse and the adjacent segment." This theorem can be used twice to probe the Pythagorean theorem and of course it is applied to special cases such as 30–60–90- and 45–45–90-degree triangles. Then we can continue this to the obtuse or acute triangle and get a similar theorem for the general triangle. The use of similar triangles opens a field of generalized theorems on the right triangle. That is the area of any figure on the hypotenuse is equal to the sum of the areas of two similar figures on the legs, let them be polygons, circles, semicircles or lunes. To find the altitude of any triangle in terms of its sides is merely a special use of the Pythagorean theorem and from the altitude it is but a short step to Hero's formula for the area in terms of the sides of a triangle.

Problems involving area can best be taught with emphasis on continuity since the basis of area is; the number of rows of squares and the number in each row. We usually think of it as the length times the width. We can use the formula $A = lw$

for all plane figures. The parallelogram can be compared by reducing it to a rectangle. The triangle can then be shown to be equal to $\frac{1}{2}$ the area of the rectangle by drawing the diagonal and showing the base the same and the altitudes equal. A trapezoid equals two such triangles which use the parallel sides for bases and having the same altitudes. In the case of the regular polygon half the apothem times the perimeter is merely a case of adding all the triangles formed by drawing radii to the center from each vertex. The area of a circle then becomes half the apothem times the perimeter or one-half pi times the diameter times the radius. This is really a special case of $A=lw$ since the triangle is based on the same. By including Hero's formula for the area of a triangle which is a special case of $A=lw$ we can find the area of any broken line figure when its sides and diagonals are given. In the case of the approximate area of the sphere the same is true since it is proportional to the area of the great circle. A similar approach can be made to all problems involving volume with the basic concept the cube or $V=lwh$.

The problem of construction with compasses and st. edge is always one of putting together in different ways a few fundamental constructions. These can be listed as: constructing the circle, carrying a line of a certain length, bisecting an angle, dropping a perpendicular from a point to a line, and constructing an angle equal to a given angle. We can consider the perpendicular to a line at a point on it as bisecting a straight angle; bisecting a line as bisecting an angle when the sides are equal and the vertex unknown; constructing parallels as constructing an angle equal to a given angle. We could continue this for all of the constructions used in plane geometry.

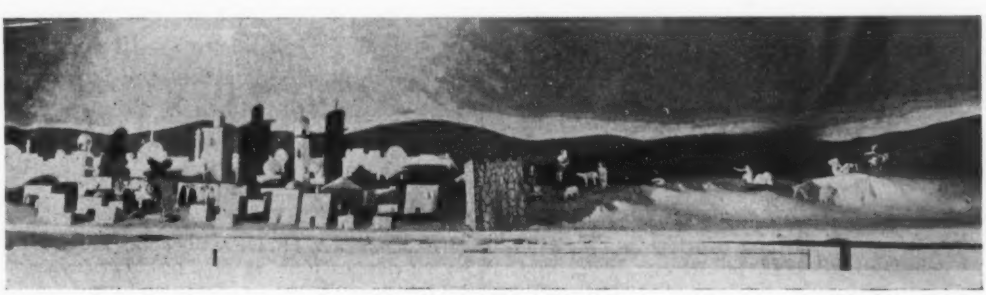
The above illustrations showing continuity in the theorems on congruency, quadrilaterals, inscribed angles, similar figures, areas and the simple constructions were not intended to be exhaustive or to be the only line of continuity running through these sets of theorems. The field of mathematics is not piecemeal, or composed of isolated bodies of knowledge therefore it is up to the teacher to show the field as a continuous one. The average student does not have a broad vision and he must be taken to the mountain top, now and then, so he can see a panoramic view of those items that appear to him as isolated facts. In this paper no plan was given for letting these illustrations lead from one into the other. This integration can be accomplished easily because no one block of knowledge is studied and then dropped. For example in most texts we study similar triangles then later the Pythagorean theorem and towards the end of the text Hero's formula. There is a continuity in this set of theorems but for other reasons they are set apart. But when each is taken we must recall what our original problem was and proceed from it. This allows a fine maintenance program, provides spaced learning, enables the student to associate the new learning with that with which he is familiar and gives him another body of knowledge where the old helps the new and the new adds meaning to the old. In mathematics the most logical of all science where the foundations are well laid and exceptions are very rare it would be a serious breach of trust if we as mathematics teachers did not help the students see the "forest" instead of just single "trees."

Mathematics and Christmas

By Kate Bell
Lewis and Clark High School, Spokane, Washington

In December, 1934 the large Christmas tree which always stood in the front hall of our high school each year was without a sponsor. The classes in solid geometry had been making some interesting models of star polyhedrons and a pupil in one of the plane geometry classes had turned in a decorated ornament based on plane five pointed stars. These two things inspired the plan to have the decorating of the tree sponsored by the mathematics department. The solid geometry class took the lead and made themselves responsible for placing the decorations on the tree and supplying the lighting as well as for making contributions of ornaments. Everyone joined in to help. Tablet backs and any pasteboard of medium weight were used in making the forms. The pupils who understood geometric construction used that technique, the younger ones used protractors and compasses. Stars of all kinds, crescents, tetrahedrons, long slim pyramids masquerading as icycles, and geometric snowmen were drawn and cut out. Pennies were collected to buy show card paint and tinsel and then the decorating began. The forms were painted with white show card paint and the tinsel was sprinkled on while the paint was wet. The addition of a little glue to the paint keeps the tinsel from dropping off and white kalsomine may be used instead of white show card paint. The tinsel is that used in lettering advertising signs and can be secured at any paint store. It is not cheap, but the use of a white paint helps, as it adds reflective power and makes an economical use of tinsel possible. Children naturally like color and so they experimented with colored tinsel and colored paint but everyone agreed that the white and silver combination was most effective. The result of this combined effort of the pupils was a gorgeous tree. A member of the faculty laughingly said that for once the faculty of the school was unanimous about something. They all agree that this was the loveliest tree the school had ever had. Occasionally since, the department has sponsored a similar tree. It would grow to be an old story if it were done every year.

However, practically every Christmas some class in the department has made a contribution to the decorations. One year a huge, sparkling blue, twenty pointed star polyhedron, with a shower of small stars hanging from its points, hung under the chandelier in the marble entrance hall, its blue blending with the blue markings of the marble. Every time the door was opened the stars swayed and sparkled, reflecting light from the lamps above.



The First Christmas
A Three Dimensional Frieze

Another year a large sparkling star polyhedron hung under each light in the front hallway. Some were based on the icosahedron and others on the dodecahedron. The solid geometry classes were always active and in 1937 devised a dioramic frieze effect to hang above the main entrance to the auditorium. They made the Christmas story their theme. The pupils worked out all the perspective for the houses in the town of Bethlehem and did all the construction work connected with the project. The electrician of the group devised a scheme for lighting the houses and the star. The whole thing was most effective.

Through the years, many classes have made Christmas card bulletin boards and show case exhibits all using the white and silver geometric decorations combined with a blue background. The blue has been found most effective to bring out the beauty of the designs used.

December 1943, furnished the real climax to these Christmas activities when the Associated Student Body asked the mathematics department to furnish some decorations for the Christmas trees at the two large military hospitals in the city. The Associated Student Body furnished the material and the department did the work. The result was the construction of over seven hundred decorations of all shapes and sizes. The pupils displayed amazing ingenuity in combining mathematical forms of all kinds. The algebra classes experimented with parabolas, the trigonometry classes with spirals and polar coordinate curves and freshman boys put their airplane modeling to use and produced as good star polyhedrons as any upperclassmen ever made. Thus the pupils vied with each other to provide Christmas cheer for the men who had sacrificed to keep our Christmas inviolate for us. These decorations are packed away at the hospitals for use again next year. The Mathematics club sponsored the reproduction of part of them for an exhibit to be sent to Teachers College, Columbia University.

All these projects have been fun; the pupils learned a great deal about mathematical construction; and the product in every case gave pleasure to all who saw it. For once the work of the mathematics department literally outshone that of all the others.

A Twelve-pointed Star Polyhedron

In order to make this star polyhedron one must follow a rather difficult, complicated procedure. First, make a core which is based on a regular dodecahedron, which in turn is based on a pentagon having an edge of one inch. Then construct a completed dodecahedron by gluing the edges of the faces together with a quick drying airplane glue. The next procedure is to construct twelve pyramids. To do this, draw a circle with a radius of three inches. Then on the circumference, mark off chords of one inch to correspond with the measurement of one edge of one pentagon. Now, with a pair of scissors, cut from the circumference line to the center so that there will be six faces. Using one of these faces for a flap, glue the two outer faces together. The pyramids will then have five faces, and the base of each pyramid should fit exactly onto a face of the completed dodecahedron. Last, glue the bases of the twelve pyramids to the twelve faces of the completed dodecahedron. It takes a great deal of patience to make a star polyhedron, but when it is finished, it will be worth one's admiration.

In renewing her subscription to The Mathematics Teacher, on November 14, 1944, Miss Alvena Louise Miller, of Algona, Iowa, said, "The November number of The Mathematics Teacher was especially fine and this subscription has been on my 'must' list for some time. I would not feel that I could teach mathematics without this journal. I trust this will reach you before the December issue is sent out."

We hope that all other members not only feel the same way, but that they will be as prompt in renewing their subscriptions.—The Editor.

The Duodecimal System

By W. C. Janes
Manhattan, Kansas

In view of the fact that there seems to be a growing tendency toward the use of decimal notation in almost all lines of numerical computation, let us examine the possibility of a scheme which might be more useful than our present one. Our present method of representing numbers is based on 10. Probably this is due to the fact that we have two hands of five digits each, and that either consciously or unconsciously we use these digits to help us in making numerical calculations. Furthermore, since the psychologists tell us that some sort of muscular reaction is associated with almost every thought process which we perform, there are people who feel that we are merely a race of finger counters and that it would be folly ever to try to adopt any base other than 10. We agree that it would be premature seriously to recommend the adoption of any other base at the present time. But there is ample evidence that the world has long realized that 10 is not always the most convenient base. The dozen has been found useful in many ways; and though the foot may be a rather arbitrary unit of length, the fact that it is divided into 12 equal parts is more likely due to the convenience of 12 than mere accident.

Let us further examine the possibilities of 12 as a base. We will find that in representing fractions in decimal form it is more convenient than 10. However, to develop our idea, let us suppose that we were children free from numerical prejudices due to training and experience. We would first learn to count. We would learn a dozen distinct symbols. These symbols would be somewhat as follows:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *, †.

We would learn to say zero, one, two, three, four, five, six, seven, eight, nine, star, dagger. To us, being free of prejudice, star and dagger would be one-digit numbers which were just as natural as any others of the set. But as a matter of fact we are somewhat startled at the suggestion that the numbers which are ordinarily called 10 and 11 should be labeled "star" and "dagger." And it is not expected that star and dagger are really to be the names used if the duodecimal system is ever adopted. They are used merely because the writer did not wish to use Greek letters; neither did he wish to coin new symbols merely by writing some well known symbols backward or upside down. The problem of adopting a suitable symbolism will fall to future generations.

Next, let us write the numbers in the "teens." But before doing so, we must be warned not to confuse the notation of our new system with the ordinary system. In our new system the number *10 means one dozen.* 11 means one more than a dozen. 12 means two more than a dozen, etc. The numbers in the teens are:

10	11	12	13	14
ten,	eleven,	twelve,	thirteen,	fourteen,

15	16	17	18
fifteen,	sixteen,	seventeen,	eighteen

19	1*	1†
nineteen,	starteen,	daggerteen.

The next number in the sequence is *20* and it *means two dozens.* We know how to count in the duodecimal system:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *, †, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1*, 1†, 20, etc.

30 represents 3 dozens.
40 represents 4 dozens.
.
.
.
90 represents 9 dozens.

90 in the new system is equivalent to an

ordinary 108. We also have "starty" (*0) and "daggerty" (†0). *0 is equivalent to an ordinary 120. †0 is equivalent to an ordinary 132. 100 is equivalent to an ordinary 144. Let us count from ninety through starty and daggerty to one hundred. The writer is somewhat embarrassed to allow the following peculiar names to appear in print, but he hopes for a tolerant attitude on the part of the reader.

90 91 92 93
ninety, ninetyone, ninetytwo, ninetythree,

94 95 96 97
ninetyfour, ninetyfive, ninetysix, ninety-

98 99 9*
seven, ninetyeight, ninetynine, ninetystar,

9† *0 *1 *2
ninetydagger, starty, startyone, startytwo,

*3 *4 *5 *6
startythree, startyfour, startyfive, starty-

*7 *8 *9
six, startyseven, startyeight, startynine,

** *† †0
startystar, startydagger, daggerty, dag-

†1 †2 †3
gertyone, daggertytwo, daggertythree, dag-

†4 †5 †6
gertyfour, daggertyfive, daggertysix,

†7 †8
daggertyseven, daggertyeight, daggerty-

†9 †* ††
nine, daggertystar, daggertydagger, one-

100
hundred.

Let us now indicate the nature of addition.

$$3+2=5$$
$$4+3=7$$
$$8+2=\ast$$
$$6+5=\dagger$$
$$8+4=10$$
$$7+6=11$$
$$9+8=15$$
$$\dagger+\ast=19$$
$$\dagger+\dagger=1\ast$$

From this we see how arithmetic would be taught in the primary grades. It would be practically as easy to learn as the present system. The child would be required to master a few more elementary combinations than at present; but this would be more than offset by the advantage gained in later work.

It is not necessary to discuss subtraction.

The nature of the following multiplication table is self-evident.

	1	2	3	4	5	6	7	8	9	*	†	10
1	1	2	3	4	5	6	7	8	9	*	†	10
2	2	4	6	8	*	10	12	14	16	18	1*	20
3	3	6	9	10	13	16	19	20	23	26	29	30
4	4	8	10	14	18	20	24	28	30	34	38	40
5	5	*	13	18	21	26	2†	34	39	42	47	50
6	6	10	16	20	26	30	36	40	46	50	56	60
7	7	12	19	24	2†	36	41	48	53	5*	65	70
8	8	14	20	28	34	40	48	54	60	68	74	80
9	9	16	23	30	39	46	53	60	69	76	83	90
*	*	18	26	34	42	50	5*	68	76	84	92	*0
†	†	1*	29	38	47	56	65	74	83	92	*1	†0
10	10	20	30	40	50	60	70	80	90	*0	†0	100

As concerns division, we are somewhat at a loss to discuss it until we thoroughly learn our multiplication table. However let us divide 476 by 2. Remember that

$$476 = 4(\text{one dozen})^2 + 7(\text{one dozen}) + 6.$$

$$2\overline{)476}$$
$$239.$$

Another example:

$$3\overline{)539}$$
$$193.$$

Having learned how to use the integers in problems dealing with addition, sub-

traction, multiplication, and division, we will turn to the problem of fractions. Since one dozen contains so many more factors than the ordinary ten, it is in the work with fractions that the duodecimal system excels. This is shown by an examination of the tables which follow. In the table immediately below, the first column contains a list of ordinary fractions many of which are very common and some of which are not easily represented as ordinary decimals; the second column shows them as common fractions in duodecimal form; and the third column shows that they are fractions of one decimal place in the new system.

Ordinary Common Fractions	Duodecimal Common Fractions	Duodecimal Equivalents
1/12	1/10	.1
1/6	1/6	.2
1/4	1/4	.3
1/3	1/3	.4
5/12	5/10	.5
1/2	1/2	.6
7/12	7/10	.7
2/3	2/3	.8
3/4	3/4	.9
5/6	5/6	.*
11/12	†/10	.†

In the above table 1/5 is conspicuous by its absence. 1/5 is represented by the repeating decimal .24972497——. 1/5 is less important than 1/3 and the writer believes that its exclusion from the new system as a simple decimal is less important than the exclusion of 1/3 from our present system. Furthermore, for many practical purposes, to approximate 1/5 as .25 would be entirely satisfactory. At least, it is more accurate to say that 1/5 = .25 than to say in the ordinary system that 1/3 = .33. In the new system we might actually be able to go over to a general use of decimals without introducing such hybrids as

$$1/3 = .33\ 1/3$$

The following self-explanatory table lists some of the commoner fractions of two decimal places.

Ordinary Common Fractions	Duodecimal Common Fractions	Duodecimal Equivalents
1/144	1/100	.01
1/72	1/60	.02
1/48	1/40	.03
1/36	1/30	.04
1/24	1/20	.06
1/18	1/16	.08
1/16	1/14	.09
1/9	1/9	.14
1/8	1/8	.16
2/9	2/9	.28
3/8	3/8	.46

We will not further illustrate the greater simplicity of the duodecimal system. As present customs and commercial procedures are not well adapted to it, we do not advise an attempt to bring about its immediate adoption. But the carpenter already learns certain tricks of his trade which seem to be equivalent to the use of 12 as a base. It is possible that an engineer familiar with the duodecimal system might prefer it provided he could get the other members of his profession to adopt it. Surely the financier and astronomer would not object to it, for it seems wiser to divide the year into 12 parts as nearly equal as possible than to attempt any other plan. The new scheme would fit into methods of measuring angles as well as our present one, and it would be adaptable to the present method of measuring time in hours, minutes and seconds.

In closing this discussion, we venture to suggest that if a committee were ever appointed to study the possibility of a uniform system of weights and measures to be used throughout the entire world with a view to the adoption of a system that would be nicely adaptable to decimal computation it might be well to investigate the possibilities of the duodecimal system.

The Formula—The Core of Algebra

By Abraham J. van Zyl
South Africa

Many teachers and textbook-writers have emphasized the importance of the formula in teaching algebra. But few have claimed that algebra is the study of formulas. Indeed it is difficult to find any algebra that can not be classified as the derivation, evaluation, manipulation or interpretation of formulas. The teacher, who fails to build his algebra course around the formula, also fails to give his pupil a clear idea of what algebra is.

The pupil should see algebra as a kind of shorthand for stating relationships, that is, formulas. He knows, for example, that the area of a rectangle can be obtained by multiplying its length by its width:

$$\text{Area} = \text{length} \times \text{width}$$

In the shorthand of algebra it becomes

$$A = lw$$

He may, however, object that this does not facilitate matters. Then teach him other formulas, for example,

$$\text{Area of a circle} = 3.14\, r^2$$

$$\text{Volume of a tennis ball} = \frac{4}{3} \times 3.14\, r^3$$

He will not be able to dispute the usefulness of such formulas.

The following are other interesting formulas which the pupil should study:

$F = \frac{9}{5}C + 32$ (to change centigrade into Fahrenheit readings)

$d = 16\, t^2$ (distance in feet that a stone will fall in t seconds)

$s = 32\, t$ (speed of stone after t seconds)

$d = \sqrt{\frac{2\,hr}{5280}}$ (distance of horizon from a point h feet above sea-level; r is the radius of the earth)

$A = C\left(1 + \frac{i}{100}\right)^n$ (amount, A, that a capital, C, will become in n years if it is invested at 1% compound interest)

From beginning to end algebra is concerned with the formula: the interpretation, evaluation, construction and manipulation of the formula. But if the teacher of algebra is to be a successful teacher he must use only real formulas, that is, formulas used in some sphere of our daily lives: in business, in geography, in farming, in engineering, in electricity, in physics, in chemistry, in any of the many branches of industry and of the pure and applied sciences.

Whenever a new formula is introduced the pupil should be given sufficient information about the field in which it is used, so that he will fully understand what he is doing and not get the feeling that he is merely juggling with the letters of the alphabet.

We often forget the importance of *purpose*. Life without *purpose* is a day to day existence which can satisfy few people. This is true for children as well as for adults. Every little deed they do must be motivated, must lead somewhere. The *Reader's Digest* of January, 1942 contained an article, "Crisis at the Kindergarten," with a most graphic description of a bright girl who was found unable to button buttons when sent to a school for superior children. She had to be taken away from the school because she was judged to be

subnormal. When her mother asked her at home why she did not button the buttons she replied: "They didn't button up anything. What's the use of just buttoning and unbuttoning?"

One often wonders what is the use of all the buttoning and unbottoning that is done in algebra from year to year: we add and subtract, multiply and divide, factorize and expand, remove brackets and insert brackets, integrate and differentiate; we simplify fractions and complicate fractions, we button and unbotton until we are mentally dazed, confused and

by the study and evaluation of formulas such as the following:

$A = lw$ (area of rectangle)

$C = 2\pi r$ (circumference of circle)

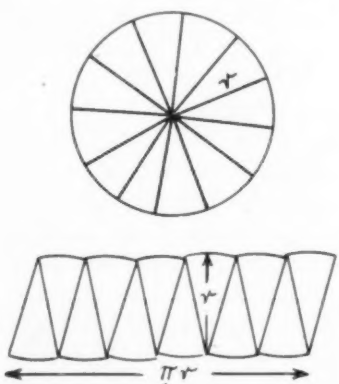
$A = bh$ (area of parallelogram)

$A = \pi r^2$ (area of circle)

$A = \dfrac{bh}{2}$ (area of triangle)

$F = \dfrac{9}{5}C + 32$ (conversion of centigrade to Fahrenheit readings)

The pupil should be required to measure several circumferences and diameters and



Area of circle
$= $ Area of parallelogram
$= \pi r \times r$
$= \pi r^2$

stupefied. And then we say: mathematics develops the mind!

There are sufficient interesting formulas if only we look around us; it is not necessary to use "nonsense" formulas as we still do so often. Let us button and unbutton those formulas which can be applied in the many different branches of industry, economics and science. At present the various branches of warfare will supply a multitude of interesting formulas.

The algebra course might be introduced

so find π, unless, of course, he has had opportunity to do it in the physical science course. The formula for the area of a circle should be found by dividing a circle into sectors, cutting them out and pasting them together to form a parallelogram as shown in the figure.

After this I would continue with the interpretation and evaluation (that is, the use of) formulas *which the beginner might find too difficult to construct himself*. Examples of such formulas are:

$A = C\left(1 + \dfrac{i}{100}\right)^n$ (compound interest)

$D = \sqrt{\dfrac{2hr}{5280}}$ (distance to horizon)

$f = \dfrac{uv}{u+v}$ (focal length of a lens)

$$S=\frac{n}{2}\,[2a+(n-1)d] \quad \text{(sum of an arithmetical series)}$$

$$S=\frac{n}{6}\,(n+1)\,(2n+1) \quad \text{(sum of the squares of the first } n \text{ natural numbers)}$$

$$V=\frac{4}{3}\pi r^3 \quad \text{(volume of a sphere)}$$

$$t=2\pi\sqrt{\frac{l}{g}} \quad \text{(time of oscillation of a pendulum)}$$

$$A=\tfrac{1}{4}\sqrt{(a+b+c)\,(b+c-a)\,(c+a-b)\,(a+b-c)} \quad \text{(area of a triangle with sides } a,\ b,\ c)$$

The pupil will be told that engineers use many formulas which they themselves cannot construct. But naturally they must know how to read and interpret such formulas. The language of algebra often is more important than the manipulation of algebraical expressions.

The reader may object that so far the course which has been outlined here hardly contains any algebra. Evaluation of formulas, that is, substituting numerical values for letters of the alphabet, is little more than arithmetic. This may be true, but at the same time the pupil will feel that he is doing something worth while, and that he is not merely buttoning and unbuttoning buttons that button up nothing in particular. At this stage he will almost be ripe for some manipulation, but I would make doubly sure of my case by first requiring him to draw the graphs of some formulas.

These will help him to understand how the different quantities represented by the symbols in the formula change under varying conditions. He will probably be familiar with the various methods of presenting statistics graphically, especially with bar and line graphs. These should be utilized to introduce him to the functional type of graph.

Only then should the pupil learn how to manipulate formulas. He will find it useful to be able to change the subject. With easy formulas such as $s=d/t$ he can use common sense methods for deriving the corresponding formulas $d=st$ and $t=d/s$, but with more difficult examples he will soon learn to use the rule "whatever you do to the left hand side of the formula (addition, subtraction, multiplication, division, etc.) do the same to the right hand side." At this stage he learns something about simple equations and their use to solve problems. Every problem, of course, in its most general form gives a formula; if the values of all the variables excepting one are known, the formula becomes an equation.

We can now briefly look at the rest of high school algebra (logarithms, brackets, indices, factorization, simplification of fractions, calculus) and see how it fits into the course as outlined above.

When evaluating a formula such as $A=C(1+i/100)^n$ and n is large, the pupil will feel the need for some quick method of calculation. This is the psychological moment for introducing logarithms. If calculating machines are available, the pupil can learn how to use them. He might even be taught the use of the slide rule.

The compound interest formula can further serve as introduction to lessons on the use of indices, of brackets and of fractions. The pupil should look upon brackets and indices as shorthand devices.

With a group of bright pupils the teacher might try to construct the compound interest formula. In connection

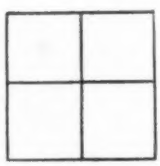
with this they will learn the relation $a^2+2ab+b^2=(a+b)^2$ and can study the binomial theorem if necessary.

The calculus is necessary for deriving one formula from another e.g. $S=32t$ from $d=16t^2$. This work will of course be related to the work on graphs and the differential coefficient of a formula can be explained as the gradient of the graph of the formula.

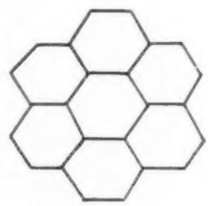
The main point is that the pupil should never get lost in pages of manipulative material without feeling that he is doing something worth while. The formula will always provide this feeling of worthwhileness. In connection with this might be pointed out the danger of misusing workbooks. By themselves many workbooks provide the worst form of buttoning and unbuttoning if used indiscriminately. But when they are used in connection with a central theme such as is provided by the formula, they become very useful teaching aids.

A problem such as the following provides an interesting example of the use of formulas: Why does the bee make its cells six-sided and not triangular, square or circular? A study of the accompanying figures and the table of formulas will show that the circle has the biggest area for a given perimeter and hence is most economical on beeswax. On the other hand space is wasted between the circular cells, hence the bee is forced to choose the hexagonal cell which gives the next highest perimeter-area ratio and wastes no space.

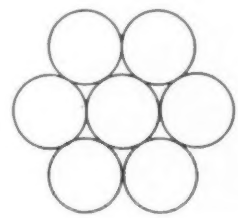
Triangles



Squares



Hexagons

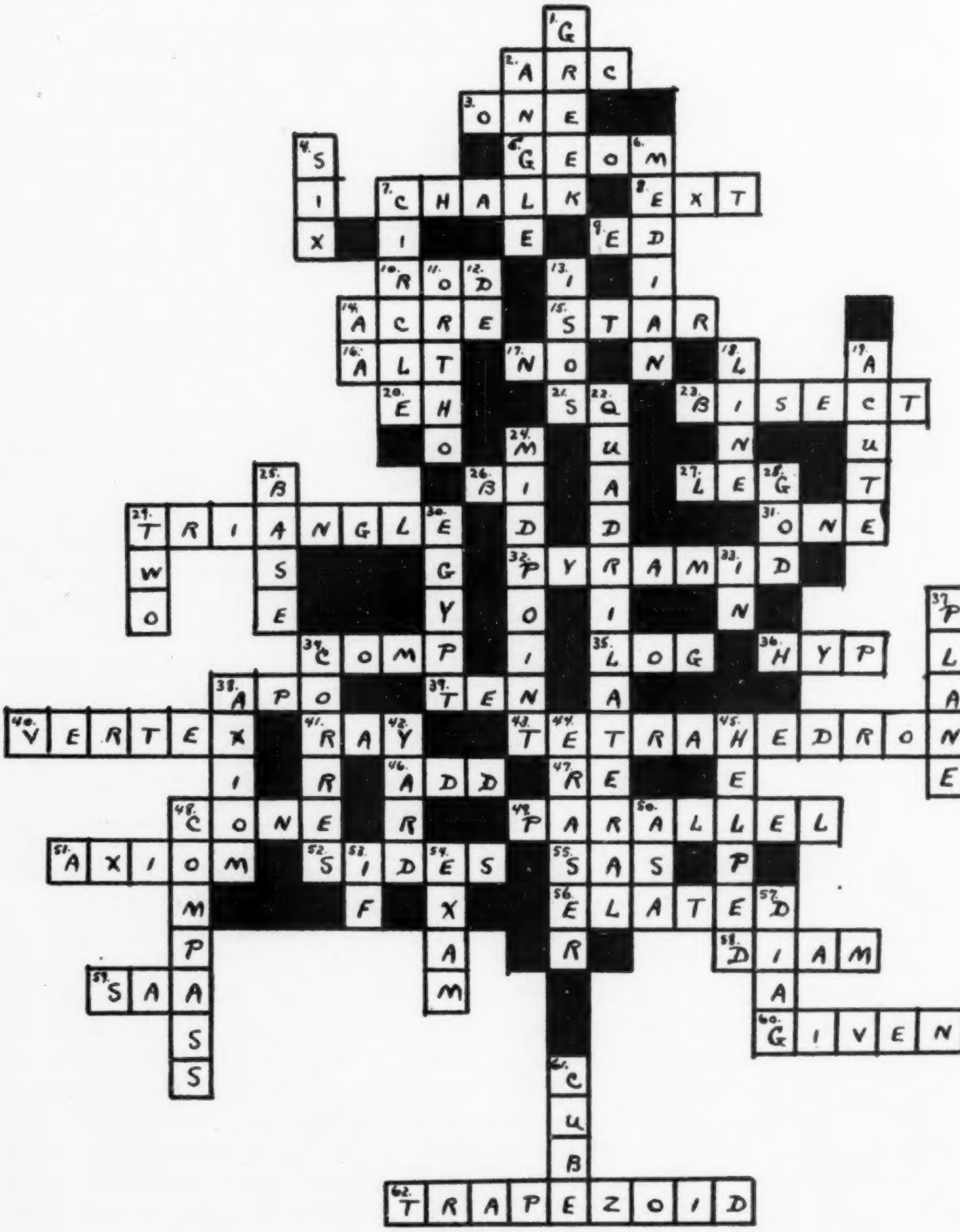


Circles

Regular figures	Triangle side t	Square side S	Hexagon side h	Circle radius r
Formula for perimeter	$3t$	$4s$	$6h$	$2\pi r$
Perimeter Formula for area	12 $\frac{\sqrt{3}t^2}{4}$	12 S^2	12 $\frac{\sqrt{27}h^2}{2}$	12 πr^2
Area	7	9	10.5	11.5

Research workers in this field have come to the conclusion that in the teaching of algebra great care should be taken not to give too much time to sheer manipulation of symbols, and that more attention should be paid to the reading and interpretation of symbols. Such manipulation as is necessary, especially in the case of the future mathematics specialist who has to free his mind for more advanced work, should as far as possible take place in connection with meaningful formulas.

Key to Crossword Puzzle on Page 338*



* Puzzle and key contributed by Miss Francis J. Bailey, Portage, Wisconsin.

THE ART OF TEACHING

School Situations Vitalize Mathematics

By Bertha Weir Palmer
Glen Rock Junior High School, Glen Rock, New Jersey

In our 9th grade course in General and Commercial Mathematics the pupils have had an opportunity to "learn to do by doing." The experiences which have arisen from the venture have made the course both profitable and enjoyable. The Glen Rock Junior High School has an enrollment of almost 300 pupils, of whom some twenty take this course in their 9th year. Much of the success of our undertaking to integrate the course with the business life of the school is due to a sympathetic and understanding principal, cooperative and friendly secretaries and the very helpful teacher in charge of the school cafeteria.

We began our integration in the school cafeteria. The pupils learned to operate and take charge of the cash register. Each pupil was justly proud when the money was counted by another member of the class and the amount agreed with the total recorded by the cash register. Adding and subtracting for a hungry and impatient line of pupils is more real and trying than doing sums on a practice pad. This year one of our best cashiers was a boy who had been exposed to eight years of arithmetic without learning to add. Spurred on by a desire to be cashier he studied his combinations until he could really add.

The need of accuracy and the necessity for careful recording were among the first lessons learned. The pupils were expected to count about thirty dollars each day. This came from two lunch periods, each of which was counted separately, recorded and then put in the cash box. At the end of the week, the total on hand plus the cash paid out for groceries had to balance with the amount recorded. At first it did not but they soon learned to be accurate and they knew why it was necessary they should be. At the end of the week, they also rolled the various coins, stamped the checks with the school stamp, made out the deposit slip and were taken with the money to the bank by the school secretary. At the bank the pupils had full charge of the money until it had been deposited. For some of the pupils it was their first visit to a bank.

The pupils also paid the cash grocery bills which necessitated their learning the value of a receipted bill. They kept a record of the bills paid by check through the school office. At the end of each month they balanced the cafeteria account and made out a statement for the school office and the cafeteria. This involved taking an inventory of the supply closet whose value ranged from \$225 to \$275 and gave some real practice in computation. Several practical situations resulted from this experience. One—When unpacking a carton of peanut butter, they found a broken jar. This had to be reported to the school office with a request that a letter be sent to the firm. The reply, that the cost of the jar be deducted from the bill, gave them an insight of how business operated.

From the cafeteria money the pupils cashed small checks for faculty members and made change for those desiring it. A counterfeit coin in the cafeteria money introduced that topic. A very interesting sound film, "Know Your Money," was obtained from the Treasury Department which was very educative and enjoyable.

The second venture resulted from an ex-

planation of the school's bookkeeping system by the school secretary. She showed the pupils how the fourteen different school accounts merged into one checking account at the bank. During the discussion that followed, it was decided that each pupil would keep a duplicate book of his own of these accounts and try to balance with the bank each month. This also involved a study of checks to see why they were valuable, safe and convenient. Each pupil kept a record of the number, date and amount for which each check was drawn. He also kept a record of the deposits. That this gave excellent practice which challenged each youngster is shown by the amount of work required every month. The number of checks written during a month varied from 24 to 51. One month the credits were $2820.00 and the expenditures were $2348.13. Each pupil who did check with the bank had a right to feel he had done a very good piece of work. Outstanding checks and bank service charge had a very real meaning to these pupils. They knew what it meant when an account was in the red. This happened each month to one or more accounts. Since the bank handled them as one account the reds were carried by the others and they in their turn helped to carry some other account. It was necessary to stop payment on one check so they had an opportunity to learn the process and the bookkeeping involved. They also had first hand experience with a certified check.

One group of these pupils returned five days after their graduation to take inventory of the cafeteria stock room and make the final cafeteria report. A second group returned ten days later to balance the school accounts for the fiscal year 1943–1944. These are but two of many incidents that occurred to show how much interest they had in their work and how well they enjoyed it. How would you rate their responsibility for seeing a task to its conclusion?

United Nations Education Kit Available to Schools

In the belief that education must play an increasing role in forging world understanding, the U. S. Office of Education has cooperated with the United Nations Information Office in the preparation of a United Nations Education Kit, John W. Studebaker, U. S. Commissioner of Education, announced today.

Teaching materials and visual aids in the kit may be used with high school or college classes as the basis for a unit of study on the United Nations. The kit may also be effectively used with adult clubs and discussion groups.

Each kit contains a reprint of "Building a United World," a study guide on the United Nations in war and peace, originally published in the September 20 edition of "Education for Victory"; 15 copies of "The United Nations Today and Tomorrow," 15 copies of "The United Nations—Peoples and Countries"; and large picture charts.

"The United Nations Today and Tomorrow" provides material for study organized under five headings: Who are the United Nations; Forerunners of the United Nations; How the United Nations Came into Being; How the United Nations Cooperate in War; and How the United Nations Cooperate in Peace. "The United Nations—Peoples and Countries" discusses each of the 37 nations in terms of geography, history, cultural uniqueness, political system, economy, and wartime role. Student activities, supplementary readings, and discussion questions are included in the "Education for Victory" reprint.

"The winning of the peace demands a citizenry trained to clarify its ideas on the problems of a rapidly shrinking world," Dr. Studebaker said. "Education for world understanding should begin in our schools."

The United Nations Education Kit may be purchased for $3.50 from the United Nations Information Office, 610 Fifth Avenue, New York 20, New York. It is estimated that a sufficient number of copies of the pamphlets are included in each kit to make use of the kit practicable for groups of approximately 30 students. Additional copies of the materials may be purchased for use with larger classes.

IN OTHER PERIODICALS

By NATHAN LAZAR
Midwood High School, Brooklyn 10, New York

The American Mathematical Monthly

August–September, 1944, vol. 51, no. 7.

1. Wylie, Jr. C. R., "Hilbert's Axioms of Plane Order," pp. 371–376.
2. Campbell, J. W., "Motion with Respect to Moving Axes," pp. 377–381.
3. Kosambi, D. D., "The Geometric Method in Mathematical Statistics," pp. 383–389.
4. Bradshaw, J. W., "More Modified Series," pp. 389–391.
5. Menger, Karl, "On the Teaching of Differential Equations," pp. 392–395.
6. Ford, L. R., "More Geographical Questions," pp. 396–398.
7. Steinberg, Barbara, "On Trisecting an Angle" (poem), p. 398.
8. War Information: University of California ESMWT Program for Industry; Notes on the Navy V-12 Program; Deferment of Mathematicians Ages 26 Through 37; Code for Students on Form 42-A Special; The AST Reserve Program; Activities of the Committee on War Training Programs (by William L. Hart); Report on Mathematics in the V-12 Program; Summary of Recommendations, pp. 415–429.

National Mathematics Magazine

October 1944, vol. 19, no. 1.

1. Wade, T. L., "Tensor Algebra and Invariants, I," pp. 3–10.
2. Lasley, Jr., J. W., "On the Classification of the Collineations in the Plane," pp. 11–20.
3. Bell, E. T., "The Golden and Platinum Proportions," pp. 20–26.
4. Harper, Floyd S., "An Experiment in Selecting Students According to Ability and Measuring Their Achievement by Common Examinations," pp. 27–32.
5. Ransom, William R., "Using Zero and One," pp. 33–35.
6. Carlson, C. S., "Note on the Teaching of Mathematical Induction," p. 36.
7. Butchart, J. H., "A Note on Newton's Theorem," p. 37.
8. Reynolds, Joseph B., "On Interpreting a Formula," pp. 37–39.

School Science and Mathematics

October 1944, vol. 44, no. 7.

1. Keso, Edward E., "The Relationship of Mathematics and Geography," pp. 598–600.
2. Jerbert, A. R., "Think of a Number," pp. 624–628.
3. Hewitt, Glenn F., "Advertising Solid Geometry with Spectacular Topics," pp. 633–636.
4. Nyberg, Joseph A., "Notes from a Mathematics Classroom," pp. 668–671.

Scripta Mathematica

December 1943, vol. 9, no. 4.

1. Bell, Eric Temple, "Sixes and Sevens," pp. 209–231.
2. Keyser, Cassius Jackson, "Man and Men," pp. 232–236.
3. Boyer, Carl B., "Pascal's Formula for the Sums of Powers of the Integers," pp. 237–244.
4. Fraenkel, Abraham Adolph, "Problems and Methods in Modern Mathematics," pp. 245–255.
5. Richards, John F. C., "A New Manuscript of a Rithmomachia," pp. 256–264.
6. Whitlock, Jr., W. P., "Rational Right Triangles with Equal Areas," pp. 265–267.
7. Curiosa, pp. 244, 268.
8. Worrell, W. H., "Jummal Notation," pp. 272–274.
9. Resnikoff, Louis A., "Jewish Calendar Calculations," pp. 274–277.
10. McCoy, John Calvin, "The Anatomy of Magic Squares," pp. 278–284.

Miscellaneous

1. Bennett, R. D., "Trends in the Amount of Mathematics and Science Taken in High School," *School Review*, 52: 406–412, September, 1944.
2. Braverman, B., "Remedial Arithmetic," *High Points*, 26: 40–44, June, 1944.
3. Bryan, N. R., "Improving the Mathematics Preparation for Specializing in Technical and Scientific Fields," *Journal of Engineering Education*, 34: 664–668, June, 1944.
4. Dorwart, Harold L., "Mathematics—Queen and Handmaiden," *School and Society*, 60: 241–243, October 14, 1944.
5. Heard, R. R., "Interesting Approach to Geometry," *School (Secondary Edition)*, 33: 54–57, September, 1944.
6. Milloy, A. L., "Next Class—Mathematics," *School (Secondary Edition)*, 33: 50–53, September, 1944.
7. Rankin Jr., W. W., "Platform for Secondary Mathematics," *Education for Victory*, 3: 26, September 4, 1944.
8. Riess, A., "Meaning of the Meaningful Teaching of Arithmetic," *Elementary School Journal*, 45: 23–32, September, 1944.
9. Zant, J. H., "Mathematics in the Army Specialized Training Program," *Journal of Engineering Education*, 34: 703–706, June, 1944.

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